Inequalities for Functions of Selfadjoint Operators on Hilbert Spaces

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ABSTRACT The main aim of this book is to present recent results concerning inequalities for continuous functions of selfadjoint operators on complex Hilbert spaces. It is intended for use by both researchers in various fields of Linear Operator Theory and Mathematical Inequalities, domains which have grown exponentially in the last decade, as well as by postgraduate students and scientists applying inequalities in their specific areas.

Contents

Preface

Linear Operator Theory in Hilbert spaces plays a central role in contemporary mathematics with numerous applications for Partial Differential Equations, in Approximation Theory, Optimization Theory, Numerical Analysis, Probability Theory & Statistics and other fields.

The main aim of this book is to present recent results concerning inequalities for continuous functions of bounded selfadjoint operators on complex Hilbert spaces.

The book is intended for use by both researchers in various fields of Linear Operator Theory and Mathematical Inequalities, domains which have grown exponentially in the last decade, as well as by postgraduate students and scientists applying inequalities in their specific areas.

In the first chapter we recall some fundamental facts concerning bounded selfadjoint operators on complex Hilbert spaces. The generalized Schwarz's inequality for positive selfadjoint operators as well as some results for the spectrum of this class of operators are presented. Then we introduce and explore the fundamental results for polynomials in a linear operator, continuous functions of selfadjoint operators as well as the step functions of selfadjoint operators. By the use of these results we then introduce the spectral decomposition of selfadjoint operators (the *Spectral Representation Theorem*) that will play a central role in the rest of the book. This result is used as a key tool in obtaining various new inequalities for continuous functions of selfadjoint operators, functions which are of bounded variation, Lipschitzian, monotonic or absolutely continuous. Another tool that will greatly simplify the error bounds provided in the book is the *Total Variation Schwarz's Inequality* for which a simple proof is offered.

The chapter is concluded with some well known operator inequalities of Jensen's type for convex and operator convex functions. Finally, some Grüss' type inequalities obtained in 1993 by Mond & Pečarić are also presented.

Jensen's type inequalities in their various settings ranging from discrete to continuous case play an important role in different branches of Modern Mathematics. A simple search in the *MathSciNet* database of the American Mathematical Society with the key words "jensen" and "inequality" in the title reveals more than 300 items intimately devoted to this famous result. However, the number of papers where this inequality is applied is a lot larger and far more difficult to find.

In the second chapter we present some recent results obtained by the author that deal with different aspects of this well research inequality than those recently reported in the book [20]. They include but are not restricted to the operator version of the Dragomir-Ionescu inequality, Slater's type inequalities for operators and its inverses, Jensen's inequality for twice differentiable functions whose second derivatives satisfy some upper and lower bounds conditions, Jensen's type inequalities for log-convex functions and for differentiable log-convex functions and their applications to Ky Fan's inequality. Finally, some Hermite-Hadamard's type inequalities for convex functions and Hermite-Hadamard's type inequalities for operator convex functions are presented as well.

The third chapter is devoted to Čebyšev and Grüss' type inequalities.

The *Čebyšev*, or in a different spelling - *Chebyshev, inequality* which compares the integral/discrete mean of the product with the product of the integral/discrete means is famous in the literature devoted to Mathematical Inequalities. It has been extended, generalized, refined etc...by many authors during the last century. A simple search utilizing either spellings and the key word "inequality" in the title in the comprehensive *MathSciNet* database produces more than 200 research articles devoted to this result.

The sister inequality due to Grüss which provides error bounds for the magnitude of the difference between the integral mean of the product and the product of the integral means has also attracted much interest since it has been discovered in 1935 with more than 180 papers published, as a simple search in the same database reveals. Far more publications have been devoted to the applications of these inequalities and an accurate picture of the impacted results in various fields of Modern Mathematics is difficult to provide.

In this chapter, however, we present only some recent results due to the author for the corresponding operator versions of these two famous inequalities. Applications for particular functions of selfadjoint operators such as the power, logarithmic and exponential functions are provided as well.

The next chapter is devoted to the Ostrowski's type inequalities. They provide sharp error estimates in approximating the value of a function

by its integral mean and can be utilized to obtain a priory error bounds for different quadrature rules in approximating the Riemann integral by different Riemann sums. They also shows, in general, that the mid-point rule provides the best approximation in the class of all Riemann sums sampled in the interior points of a given partition.

As revealed by a simple search in *MathSciNet* with the key words "Ostrowski" and "inequality" in the title, an exponential evolution of research papers devoted to this result has been registered in the last decade. There are now at least 280 papers that can be found by performing the above search. Numerous extensions, generalizations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Probability Theory and other fields have been also given.

In this chapter we present some recent results obtained by the author in extending Ostrowski inequality in various directions for continuous functions of selfadjoint operators in complex Hilbert spaces. Applications for mid-point inequalities and some elementary functions of operators such as the power function, the logarithmic and exponential functions are provided as well.

From a complementary viewpoint to Ostrowski/mid-point inequalities, trapezoidal type inequality provide a priory error bounds in approximating the Riemann integral by a (generalized) trapezoidal formula.

Just like in the case of Ostrowski's inequality the development of these kind of results have registered a sharp growth in the last decade with more than 50 papers published, as one can easily asses this by performing a search with the key word "trapezoid" and "inequality" in the title of the papers reviewed by *MathSciNet*.

Numerous extensions, generalizations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Probability Theory and other fields have been also given.

In chapter five we present some recent results obtained by the author in extending trapezoidal type inequality in various directions for continuous functions of selfadjoint operators in complex Hilbert spaces. Applications for some elementary functions of operators are provided as well.

In approximating n-time differentiable functions around a point, perhaps the classical Taylor's expansion is one of the simplest and most convenient and elegant methods that has been employed in the development of Mathematics for the last three centuries.

In the sixth and last chapter of the book, we present some error bounds in approximating n-time differentiable functions of selfadjoint operators by

the use of operator Taylor's type expansions around a point or two points from its spectrum for which the remainder is known in an integral form. Some applications for elementary functions including the exponential and logarithmic functions are provided as well.

For the sake of completeness, all the results presented are completely proved and the original references where they have been firstly obtained are mentioned. The chapters are followed by the list of references used therein and therefore are relatively independent and can be read separately.

The Author[1]

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[1] This book is dedicated to my beloved children Sergiu & Camelia and granddaughter Sienna Clarisse.

1
Functions of Selfadjoint Operators in Hilbert Spaces

1.1 Introduction

In this introductory chapter we recall some fundamental facts concerning bounded selfadjoint operators on complex Hilbert spaces. Since all the operators considered in this book are supposed to be bounded, we no longer mention this but understand it implicitly.

The generalized Schwarz's inequality for positive selfadjoint operators as well as some results for the spectrum of this class of operators are presented. Then we introduce and explore the fundamental results for polynomials in a linear operator, continuous functions of selfadjoint operators as well as the step functions of selfadjoint operators. By the use of these results we then introduce the spectral decomposition of selfadjoint operators (the *Spectral Representation Theorem*) that will play a central role in the rest of the book. This result is used as a key tool in obtaining various new inequalities for continuous functions of selfadjoint operators which are of bounded variation, Lipschitzian, monotonic or absolutely continuous. Another tool that will greatly simplify the error bounds provided in the book is the *Total Variation Schwarz's Inequality* for which a simple proof is offered.

The chapter is concluded with some well known operator inequalities of Jensen's type for convex and operator convex functions. More results in this spirit can be found in the recent book [1].

Finally, some Grüss' type inequalities obtained in 1993 by Mond & Pečarić are also presented. They are developed extensively in a special

chapter later in the book where some applications in relation with classical power operator inequalities are provided as well.

1.2   Bounded Selfadjoint Operators

1.2.1   Operator Order

Let $(H; \langle ., . \rangle)$ be a Hilbert space over the complex numbers field $\mathbb{C}$.

A bounded linear operator A defined on H is *selfadjoint*, i.e., $A = A^*$ if and only if $\langle Ax, x \rangle \in \mathbb{R}$ for all $x \in H$ and if A is selfadjoint, then

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x \rangle| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle|. \tag{1.1}$$

We assume in what follows that all operators are bounded on defined on the whole Hilbert space H. We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators defined on H.

Definition 1 *Let A and B be selfadjoint operators on H. Then $A \leq B$ (A is less or equal to B) or, equivalently, $B \geq A$ if $\langle Ax, x \rangle \leq \langle Bx, x \rangle$ for all $x \in H$. In particular, A is called positive if $A \geq 0$.*

It is well known that for any operator $A \in \mathcal{B}(H)$ the composite operators A^*A and AA^* are positive selfadjoint operators on H. However, the operators A^*A and AA^* are not comparable with each other in general.

The following result concerning the operator order holds (see for instance [2, p. 220]):

Theorem 2 *Let $A, B, C \in \mathcal{B}(H)$ be selfadjoint operators and let $\alpha, \beta \in \mathbb{R}$. Then*

1. $A \leq A$;

2. *If $A \leq B$ and $B \leq C$, then $A \leq C$;*

3. *If $A \leq B$ and $B \leq A$, then $A = B$;*

4. *If $A \leq B$ and $\alpha \geq 0$, then*

$$A + C \leq B + C, \alpha A \leq \alpha B, -A \geq -B;$$

5. *If $\alpha \leq \beta$, then $\alpha A \leq \beta A$.*

The following *generalization of Schwarz's inequality* for positive selfadjoint operators A holds:

$$|\langle Ax, y \rangle|^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle \tag{1.2}$$

for any $x, y \in H$.

The following inequality is of interest as well, see [2, p. 221]

Theorem 3 *Let A be a positive selfadjoint operator on H. Then*

$$\|Ax\|^2 \leq \|A\| \langle Ax, x \rangle \qquad (1.3)$$

for any $x \in H$.

Theorem 4 *Let $A_n, B \in \mathcal{B}(H)$ with $n \geq 1$ be selfadjoint operators with the property that*

$$A_1 \leq A_2 \leq ... \leq A_n \leq ... \leq B.$$

Then there exists a bounded selfadjoint operator A defined on H such that

$$A_n \leq A \leq B \text{ for all } n \geq 1$$

and

$$\lim_{n \to \infty} A_n x = Ax \text{ for all } x \in H.$$

An analogous assertion holds if the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing and bounded below.

Definition 5 *We say that a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(H)$ converges strongly to an operator $A \in \mathcal{B}(H)$, called the strong limit of the sequence $\{A_n\}_{n=1}^{\infty}$ and we denote this by $(s)\lim_{n \to \infty} A_n = A$, if $\lim_{n \to \infty} A_n x = Ax$ for all $x \in H$.*

The convergence in norm, i.e. $\lim_{n \to \infty} \|A_n - A\| = 0$ will be called the *"uniform convergence"* as opposed to strong convergence. We denote $\lim_{n \to \infty} A_n = A$ for the convergence in norm. From the inequality

$$\|A_m x - A_n x\| \leq \|A_m - A_n\| \|x\|$$

that holds for all n, m and $x \in H$ it follows that uniform convergence of the sequence $\{A_n\}_{n=1}^{\infty}$ to A implies strong convergence of $\{A_n\}_{n=1}^{\infty}$ to A. However, the converse of this assertion is false.

It is also possible to introduce yet another concept of *"weak convergence"* in $\mathcal{B}(H)$ by defining $(w)\lim_{n \to \infty} A_n = A$ if and only if $\lim_{n \to \infty} \langle A_n x, y \rangle = \langle Ax, y \rangle$ for all $x, y \in H$.

The following result holds (see [2, p. 225]):

Theorem 6 *Let A be a bounded selfadjoint operator on H. Then*

$$\alpha_1 \quad : \quad = \inf_{\|x\|=1} \langle Ax, x \rangle = \max \{\alpha \in \mathbb{R} \,|\alpha I \leq A\};$$

$$\alpha_2 \quad : \quad = \sup_{\|x\|=1} \langle Ax, x \rangle = \min \{\alpha \in \mathbb{R} \,|A \leq \alpha I\};$$

and

$$\|A\| = \max \{|\alpha_1|, |\alpha_2|\}.$$

Moreover, if $Sp(A)$ denotes the spectrum of A, then $\alpha_1, \alpha_2 \in Sp(A)$ and $Sp(A) \subset [\alpha_1, \alpha_2].$

Remark 7 *We remark that, if A, α_1, α_2 are as above, then obviously*

$$
\begin{aligned}
\alpha_1 &= \min\{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A); \\
\alpha_2 &= \max\{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A); \\
\|A\| &= \max\{|\lambda| \,|\, \lambda \in Sp(A)\}.
\end{aligned}
$$

We also observe that

1. *A is positive iff $\alpha_1 \geq 0$;*

2. *A is positive and invertible iff $\alpha_1 > 0$;*

3. *If $\alpha_1 > 0$, then A^{-1} is a positive selfadjoint operator and $\min Sp(A^{-1}) = \alpha_2^{-1}, \max Sp(A^{-1}) = \alpha_1^{-1}$.*

1.3   Continuous Functions of Selfadjoint Operators

1.3.1   *Polynomials in a Bounded Operator*

For two functions $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ we adhere to the canonical notation:

$$
\begin{aligned}
(\varphi + \psi)(s) &:= \varphi(s) + \psi(s), \\
(\lambda\varphi)(s) &:= \lambda\varphi(s), \\
(\varphi\psi)(s) &:= \varphi(s)\,\psi(s)
\end{aligned}
$$

for sum, scalar multiple and product of these functions. We denote by $\bar{\varphi}(s)$ the complex conjugate of $\varphi(s)$.

As a first class of functions we consider the algebra $\mathcal{P}$ of all polynomials in one variable with complex coefficients, namely

$$
\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k \,\Big|\, n \geq 0, \alpha_k \in \mathbb{C}, 0 \leq k \leq n \right\}.
$$

Theorem 8 *Let $A \in \mathcal{B}(H)$ and for $\varphi(s) := \sum_{k=0}^{n} \alpha_k s^k \in \mathcal{P}$ define $\varphi(A) := \sum_{k=0}^{n} \alpha_k A^k \in \mathcal{B}(H) \left(A^0 = I\right)$ and $\bar{\varphi}(A) := \sum_{k=0}^{n} \bar{\alpha}_k (A^*)^k \in \mathcal{B}(H)$. Then the mapping $\varphi(s) \to \varphi(A)$ has the following properties*

a) *$(\varphi + \psi)(A) = \varphi(A) + \psi(A)$;*

b) *$(\lambda\varphi)(A) = \lambda\varphi(A)$;*

c) *$(\varphi\psi)(A) = \varphi(A)\,\psi(A)$;*

d) *$[\varphi(A)]^* = \bar{\varphi}(A)$.*

Note that $\varphi(A)\psi(A) = \psi(A)\varphi(A)$ and the constant polynomial $\varphi(s) = \alpha_0$ is mapped into the operator.

Recall that, a mapping $a \rightarrow a'$ of an algebra $\mathcal{U}$ into an algebra $\mathcal{U}'$ is called a *homomorphism* if it has the properties

a) $(a+b)' = a' + b'$;

b) $(\lambda\varphi)' = \lambda a'$;

c) $(ab)' = a'b'$.

With this terminology, Theorem 8 asserts that the mapping which associates with any polynomial $\varphi(s)$ the operator $\varphi(A)$ is a homomorphism of $\mathcal{P}$ into $\mathcal{B}(H)$ satisfying the additional property d).

The following result provides a connection between the spectrum of A and the spectrum of the operator $\varphi(A)$.

Theorem 9 *If $A \in \mathcal{B}(H)$ and $\varphi \in \mathcal{P}$, then $Sp(\varphi(A)) = \varphi(Sp(A))$.*

Corollary 10 *If $A \in \mathcal{B}(H)$ is selfadjoint and the polynomial $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint and*

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}. \tag{1.4}$$

Remark 11 *If $A \in \mathcal{B}(H)$ and $\varphi \in \mathcal{P}$, then*

1. *$\varphi(A)$ is invertible iff $\varphi(\lambda) \neq 0$ for all $\lambda \in Sp(A)$;*

2. *If $\varphi(A)$ is invertible, then $Sp\left(\varphi(A)^{-1}\right) = \left\{\varphi(\lambda)^{-1}, \lambda \in Sp(A)\right\}$.*

1.3.2   Continuous Functions of Selfadjoint Operators

Assume that A is a bounded selfadjoint operator on the Hilbert space H. If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A = \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If φ is continuous, in particular if φ is a polynomial, then the supremum is actually assumed for some points in $Sp(A)$ which is compact. Therefore the supremum may then be written as a maximum and the formula (1.4) can be written in the form $\|\varphi(A)\| = \|\varphi\|_A$.

Consider $\mathcal{C}(\mathbb{R})$ the algebra of all continuous complex valued functions defined on $\mathbb{R}$. The following fundamental result for continuous functional calculus holds, see for instance [2, p. 232]:

Theorem 12 *If A is a bounded selfadjoint operator on the Hilbert space H and $\varphi \in \mathcal{C}(\mathbb{R})$, then there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ with the property that whenever $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{P}$ such that $\lim_{n\to\infty} \|\varphi - \varphi_n\|_A = 0$,*

then $\varphi(A) = \lim_{n\to\infty} \varphi_n(A)$. The mapping $\varphi \to \varphi(A)$ is a homomorphism of the algebra $\mathcal{C}(\mathbb{R})$ into $\mathcal{B}(H)$ with the additional properties $[\varphi(A)]^* = \bar{\varphi}(A)$ and $\|\varphi(A)\| \leq 2\|\varphi\|_A$. Moreover, $\varphi(A)$ is a normal operator, i.e. $[\varphi(A)]^* \varphi(A) = \varphi(A)[\varphi(A)]^*$. If φ is real-valued, then $\varphi(A)$ is selfadjoint.

As examples we notice that, if $A \in \mathcal{B}(H)$ is selfadjoint and $\varphi(s) = e^{is}, s \in \mathbb{R}$ then

$$e^{iA} = \sum_{k=0}^{\infty} \frac{1}{k!}(iA)^k.$$

Moreover, e^{iA} is a unitary operator and its inverse is the operator

$$\left(e^{iA}\right)^* = e^{-iA} = \sum_{k=0}^{\infty} \frac{1}{k!}(-iA)^k.$$

Now, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $A \in \mathcal{B}(H)$ is selfadjoint and $\varphi(s) = \frac{1}{s-\lambda} \in \mathcal{C}(\mathbb{R})$, then $\varphi(A) = (A - \lambda I)^{-1}$.

If the selfadjoint operator $A \in \mathcal{B}(H)$ and the functions $\varphi, \psi \in \mathcal{C}(\mathbb{R})$ are given, then we obtain the commutativity property $\varphi(A)\psi(A) = \psi(A)\varphi(A)$. This property can be extended for another operator as follows, see for instance [2, p. 235]:

Theorem 13 *Assume that $A \in \mathcal{B}(H)$ and the function $\varphi \in \mathcal{C}(\mathbb{R})$ are given. If $B \in \mathcal{B}(H)$ is such that $AB = BA$, then $\varphi(A)B = B\varphi(A)$.*

The next result extends Theorem 9 to the case of continuous functions, see for instance [2, p. 235]:

Theorem 14 *If A is abounded selfadjoint operator on the Hilbert space H and φ is continuous, then $Sp(\varphi(A)) = \varphi(Sp(A))$.*

As a consequence of this result we have:

Corollary 15 *With the assumptions in Theorem 14 we have:*

    a) The operator $\varphi(A)$ is selfadjoint iff $\varphi(\lambda) \in \mathbb{R}$ for all $\lambda \in Sp(A)$;

    b) The operator $\varphi(A)$ is unitary iff $|\varphi(\lambda)| = 1$ for all $\lambda \in Sp(A)$;

    c) The operator $\varphi(A)$ is invertible iff $\varphi(\lambda) \neq 0$ for all $\lambda \in Sp(A)$;

    d) If $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$.

In order to develop inequalities for functions of selfadjoint operators we need the following result, see for instance [2, p. 240]:

Theorem 16 *Let A be a bounded selfadjoint operator on the Hilbert space H. The homomorphism $\varphi \to \varphi(A)$ of $\mathcal{C}(\mathbb{R})$ into $\mathcal{B}(H)$ is order preserving, meaning that, if $\varphi, \psi \in \mathcal{C}(\mathbb{R})$ are real valued on $Sp(A)$ and $\varphi(\lambda) \geq \psi(\lambda)$ for any $\lambda \in Sp(A)$, then*

$$\varphi(A) \geq \psi(A) \text{ in the operator order of } \mathcal{B}(H). \tag{P}$$

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [2, p. 240]:

Theorem 17 *If the operator $A \in \mathcal{B}(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$. If A is invertible, then so is B.*

If $A \in \mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A| := \sqrt{A^*A}$.

Analogously to the familiar factorization of a complex number

$$\xi = |\xi|\, e^{i \arg \xi}$$

a bounded normal operator on H may be written as a commutative product of a positive selfadjoint operator, representing its absolute value, and a unitary operator, representing the factor of absolute value one.

In fact, the following more general result holds, see for instance [2, p. 241]:

Theorem 18 *For every bounded linear operator A on H, there exists a positive selfadjoint operator $B = |A| \in \mathcal{B}(H)$ and an isometric operator C with the domain $\mathcal{D}_C = \overline{B(H)}$ and range $\mathcal{R}_C = C(\mathcal{D}_C) = \overline{A(H)}$ such that $A = CB$.*

In particular, we have:

Corollary 19 *If the operator $A \in \mathcal{B}(H)$ is normal, then there exists a positive selfadjoint operator $B = |A| \in \mathcal{B}(H)$ and a unitary operator C such that $A = BC = CB$. Moreover, if A is invertible, then B and C are uniquely determined by these requirements.*

Remark 20 *Now, suppose that $A = CB$ where $B \in \mathcal{B}(H)$ is a positive selfadjoint operator and C is an isometric operator. Then*

a) *$B = \sqrt{A^*A}$; consequently B is uniquely determined by the stated requirements;*

b) *C is uniquely determined by the stated requirements iff A is one-to-one.*

1.4   Step Functions of Selfadjoint Operators

Let A be a bonded selfadjoint operator on the Hilbert space H. We intend to extend the order preserving homomorphism $\varphi \to \varphi(A)$ of the algebra $\mathcal{C}(\mathbb{R})$ of continuous functions φ defined on $\mathbb{R}$ into $\mathcal{B}(H)$, restricted now to real-valued functions, to a larger domain, namely an algebra of functions containing the "step functions" $\varphi_\lambda, \lambda \in \mathbb{R}$, defined by

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Observe that $\overline{\varphi}_\lambda(s) = \varphi_\lambda(s)$ and $\varphi_\lambda^2(s) = \varphi_\lambda(s)$ which will imply that $[\varphi_\lambda(A)]^* = \varphi_\lambda(A)$ and $[\varphi_\lambda(A)]^2 = \varphi_\lambda(A)$, i.e. $\varphi_\lambda(A)$ will then be a projection. However, since the function φ_λ cannot be approximated uniformly by continuous functions on any interval containing λ, then, in general, there is no way to define an operator $\varphi_\lambda(A)$ as a uniform limit of operators $\varphi_{\lambda,n}(A)$ with $\varphi_{\lambda,n} \in \mathcal{C}(\mathbb{R})$.

The uniform limit of operators can be relaxed to the concept of strong limit of operators (see Definition 5) in order to define the operator $\varphi_\lambda(A)$. In order to do that, observe that the function φ_λ may be obtained as a pointwise limit of a decreasing sequence of real-valued continuous functions $\varphi_{\lambda,n}$ defined by

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ 1 - n(s - \lambda), & \text{for } \lambda \le s \le \lambda + 1/n \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

By Theorem 4 we observe that the sequence of corresponding selfadjoint operators $\varphi_{\lambda,n}(A)$ is nondecreasing and bounded below by zero in the operator order of $\mathcal{B}(H)$. It therefore converges strongly to some bounded selfadjoint operator $\varphi_\lambda(A)$ on H, see [2, p. 244].

To provide a formal presentation of the above, we need the following definition.

Definition 21 *A real-valued function φ on $\mathbb{R}$ is called upper semi-continuous if it is a pointwise limit of a non-increasing sequence of continuous real-valued functions on $\mathbb{R}$.*

We observe that it can be shown that a real-valued functions φ on $\mathbb{R}$ is upper semi-continuous iff for every $s_0 \in \mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varphi(s) < \varphi(s_0) + \varepsilon \text{ for all } s \in (s_0 - \delta, s_0 + \delta).$$

We can introduce now the operator $\varphi(A)$ as follows, see for instance [2, p. 245]:

Theorem 22 *Let A be a bonded selfadjoint operator on the Hilbert space H and let φ be a nonnegative upper semi-continuous function on $\mathbb{R}$. Then there exists a unique positive selfadjoint operator $\varphi(A)$ such that whenever $\{\varphi_n\}_{n=1}^{\infty}$ is any non-increasing sequence of non-negative functions in $C(\mathbb{R})$, pointwise converging to φ on $Sp(A)$, then $\varphi(A) = (s)\lim \varphi_n(A)$.*

If φ is continuous, then the operator $\varphi(A)$ defined by Theorem 12 coincides with the one defined by Theorem 22.

Theorem 23 *Let $A \in \mathcal{B}(H)$ be selfadjoint, let φ and ψ be non-negative upper semi-continuous functions on $\mathbb{R}$, and let $\alpha > 0$ be given. Then the functions $\varphi + \psi$, $\alpha\varphi$ and $\varphi\psi$ are non-negative upper semi-continuous and $(\varphi + \psi)(A) = \varphi(A) + \psi(A)$, $(\alpha\varphi)(A) = \alpha\varphi(A)$ and $(\varphi\psi)(A) = \varphi(A)\psi(A)$. Moreover, if $\varphi(s) \le \psi(s)$ for all $s \in Sp(A)$ then $\varphi(A) \le \psi(A)$.*

We enlarge the class of non-negative upper semi-continuous functions to an algebra by defining $\mathcal{R}(\mathbb{R})$ as the set of all functions $\varphi = \varphi_1 - \varphi_2$ where φ_1, φ_2 are nonnegative and upper semi-continuous functions defined on $\mathbb{R}$. It is easy to see that $\mathcal{R}(\mathbb{R})$ endowed with pointwise sum, scalar multiple and product is an algebra.

The following result concerning functions of operators $\varphi(A)$ with $\varphi \in \mathcal{R}(\mathbb{R})$ can be stated, see for instance [2, p. 249-p. 250]:

Theorem 24 *Let $A \in \mathcal{B}(H)$ be selfadjoint and let $\varphi \in \mathcal{R}(\mathbb{R})$. Then there exists a unique selfadjoint operator $\varphi(A) \in \mathcal{B}(H)$ such that if $\varphi = \varphi_1 - \varphi_2$ where φ_1, φ_2 are nonnegative and upper semi-continuous functions defined on $\mathbb{R}$, then $\varphi(A) = \varphi_1(A) - \varphi_2(A)$. The mapping $\varphi \to \varphi(A)$ is a homomorphism of $\mathcal{R}(\mathbb{R})$ into $\mathcal{B}(H)$ which is order preserving in the following sense: if $\varphi, \psi \in \mathcal{R}(\mathbb{R})$ with the property that $\varphi(s) \le \psi(s)$ for any $s \in Sp(A)$, then $\varphi(A) \le \psi(A)$. Moreover, if $B \in \mathcal{B}(H)$ satisfies the commutativity condition $AB = BA$, then $\varphi(A)B = B\varphi(A)$.*

1.5 The Spectral Decomposition of Selfadjoint Operators

Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda (A) \tag{1.5}$$

is a projection which reduces A.

The properties of these projections are summed up in the following fundamental result concerning the spectral decomposition of bounded selfadjoint operators in Hilbert spaces, see for instance [2, p. 256]

Theorem 25 (Spectral Representation Theorem) *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min \{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max \{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \leq E_{\lambda'}$ *for* $\lambda \leq \lambda'$;

b) $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;

c) *We have the representation*

$$A = \int_{m-0}^M \lambda dE_\lambda. \tag{1.6}$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi (A) - \sum_{k=1}^n \varphi (\lambda'_k) \left[E_{\lambda_k} - E_{\lambda_{k-1}} \right] \right\| \leq \varepsilon \tag{1.7}$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \\ \lambda'_k \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases} \tag{1.8}$$

this means that

$$\varphi (A) = \int_{m-0}^M \varphi (\lambda) \, dE_\lambda, \tag{1.9}$$

where the integral is of Riemann-Stieltjes type.

Corollary 26 *With the assumptions of Theorem 25 for A, E_λ and φ we have the representations*

$$\varphi (A) x = \int_{m-0}^M \varphi (\lambda) \, dE_\lambda x \quad \text{for all } x \in H \tag{1.10}$$

and

$$\langle \varphi (A) x, y \rangle = \int_{m-0}^{M} \varphi (\lambda) \, d \langle E_{\lambda} x, y \rangle \quad \text{for all } x, y \in H. \tag{1.11}$$

In particular,

$$\langle \varphi (A) x, x \rangle = \int_{m-0}^{M} \varphi (\lambda) \, d \langle E_{\lambda} x, x \rangle \quad \text{for all } x \in H. \tag{1.12}$$

Moreover, we have the equality

$$\| \varphi (A) x \|^{2} = \int_{m-0}^{M} |\varphi (\lambda)|^{2} \, d \| E_{\lambda} x \|^{2} \quad \text{for all } x \in H. \tag{1.13}$$

The next result shows that it is legitimate to talk about "*the*" spectral family of the bounded selfadjoint operator A since it is uniquely determined by the requirements a), b) and c) in Theorem 25, see for instance [2, p. 258]:

Theorem 27 *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min Sp (A)$ and $M = \max Sp (A)$. If $\{F_{\lambda}\}_{\lambda \in \mathbb{R}}$ is a family of projections satisfying the requirements a), b) and c) in Theorem 25, then $F_{\lambda} = E_{\lambda}$ for all $\lambda \in \mathbb{R}$ where E_{λ} is defined by (1.5).*

By the above two theorems, the spectral family $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$ uniquely determines and in turn is uniquely determined by the bounded selfadjoint operator A. The spectral family also reflects in a direct way the properties of the operator A as follows, see [2, p. 263-p.266]

Theorem 28 *Let $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. If B is a bounded linear operator on H, then $AB = BA$ iff $E_{\lambda}B = BE_{\lambda}$ for all $\lambda \in \mathbb{R}$. In particular $E_{\lambda}A = AE_{\lambda}$ for all $\lambda \in \mathbb{R}$.*

Theorem 29 *Let $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A and $\mu \in \mathbb{R}$. Then*

a) *μ is a regular value of A, i.e., $A - \mu I$ is invertible iff there exists a $\theta > 0$ such that $E_{\mu-\theta} = E_{\mu+\theta}$;*

b) *$\mu \in Sp (A)$ iff $E_{\mu-\theta} < E_{\mu+\theta}$ for all $\theta > 0$;*

c) *μ is an eigenvalue of A iff $E_{\mu-0} < E_{\mu}$.*

The following result will play a key role in many results concerning inequalities for bounded selfadjoint operators in Hilbert spaces. Since we were not able to locate it in the literature, we will provide here a complete proof:

Theorem 30 (Total Variation Schwarz's Inequality) *Let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A and let $m = \min Sp(A)$ and $M = \max Sp(A)$. Then for any $x, y \in H$ the function $\lambda \to \langle E_\lambda x, y \rangle$ is of bounded variation on $[m, M]$ and we have the inequality*

$$\left| \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \right| \leq \|x\| \, \|y\| , \qquad \text{(TVSI)}$$

where $\bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$ denotes the total variation of the function $\langle E_{(\cdot)} x, y \rangle$ on $[m, M]$.

Proof. If P is a nonnegative selfadjoint operator on H, i.e., $\langle Px, x \rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle , \qquad (1.14)$$

for any $x, y \in H$.

Now, if $d : m = t_0 < t_1 < ... < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m, M]$, then we have by Schwarz's inequality for nonnegative operators (1.14) that

$$\bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \qquad (1.15)$$

$$= \sup_d \left\{ \sum_{i=0}^{n-1} \left| \langle \left(E_{t_{i+1}} - E_{t_i} \right) x, y \rangle \right| \right\}$$

$$\leq \sup_d \left\{ \sum_{i=0}^{n-1} \left[\langle \left(E_{t_{i+1}} - E_{t_i} \right) x, x \rangle^{1/2} \langle \left(E_{t_{i+1}} - E_{t_i} \right) y, y \rangle^{1/2} \right] \right\} := I.$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$I \qquad (1.16)$$

$$\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \langle \left(E_{t_{i+1}} - E_{t_i} \right) x, x \rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \langle \left(E_{t_{i+1}} - E_{t_i} \right) y, y \rangle \right]^{1/2} \right\}$$

$$\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \langle \left(E_{t_{i+1}} - E_{t_i} \right) x, x \rangle \right]^{1/2} \sup_d \left[\sum_{i=0}^{n-1} \langle \left(E_{t_{i+1}} - E_{t_i} \right) y, y \rangle \right]^{1/2} \right\}$$

$$= \left[\bigvee_m^M \left(\langle E_{(\cdot)} x, x \rangle \right) \right]^{1/2} \left[\bigvee_m^M \left(\langle E_{(\cdot)} y, y \rangle \right) \right]^{1/2} = \|x\| \, \|y\|$$

for any $x, y \in H$.

On making use of (1.15) and (1.16) we deduce the desired result (TVSI). ∎

1.6   Jensen's Type Inequalities

1.6.1   Jensen's Inequality.

The following result that provides an operator version for the *Jensen inequality* is due to Mond & Pečarić [5] (see also [1, p. 5]):

Theorem 31 (Mond- Pečarić, 1993, [5]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 31 we have the following *Hölder-McCarthy inequality*:

Theorem 32 (Hölder-McCarthy, 1967, [3]) *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;

(ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;

(iii) If A is invertible, then $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r < 0$ and $x \in H$ with $\|x\| = 1$.

The following theorem is a multiple operator version of Theorem 31 (see for instance [1, p. 5]):

Theorem 33 (Furuta-Mićić-Pečarić-Seo, 2005, [1]) *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $m < M$ and $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. If f is a convex function on $[m, M]$, then*

$$f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) \leq \sum_{j=1}^{n} \langle f(A_j)x_j, x_j \rangle. \tag{1.17}$$

The following particular case is of interest.

Corollary 34 *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $m < M$. If $p_j \geq 0$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right) \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle, \tag{1.18}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Follows from Theorem 33 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$ where $x \in H$ with $\|x\| = 1$. ∎

Remark 35 *The above inequality can be used to produce some norm inequalities for the sum of positive operators in the case when the convex function f is nonnegative and monotonic nondecreasing on $[0, M]$. Namely, we have:*

$$f\left(\left\|\sum_{j=1}^{n} p_j A_j\right\|\right) \le \left\|\sum_{j=1}^{n} p_j f(A_j)\right\|. \tag{1.19}$$

The inequality (1.19) reverses if the function is concave on $[0, M]$.
  As particular cases we can state the following inequalities:

$$\left\|\sum_{j=1}^{n} p_j A_j\right\|^{p} \le \left\|\sum_{j=1}^{n} p_j A_j^{p}\right\|, \tag{1.20}$$

for $p \ge 1$ and

$$\left\|\sum_{j=1}^{n} p_j A_j\right\|^{p} \ge \left\|\sum_{j=1}^{n} p_j A_j^{p}\right\| \tag{1.21}$$

for $0 < p < 1$.
  If A_j are positive definite for each $j \in \{1, \ldots, n\}$ then (1.20) also holds for $p < 0$.
  If one uses the inequality (1.19) for the exponential function, that one obtains the inequality

$$\exp\left(\left\|\sum_{j=1}^{n} p_j A_j\right\|\right) \le \left\|\sum_{j=1}^{n} p_j \exp(A_j)\right\|, \tag{1.22}$$

where A_j are positive operators for each $j \in \{1, \ldots, n\}$.

1.6.2   Reverses of Jensen's Inequality

In Section 2.4 of the monograph [1] there are numerous interesting converses of the Jensen's type inequality (1.17) from which we would like to mention only two of the simplest.

  The following result is an operator version of the well known Lah-Ribarić's reverse of the Jensen inequality for real functions of a real variable, see for instance [1]:

Theorem 36 *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $m < M$ and $x_j \in H, j \in \{1, \ldots, n\}$ with*

$\sum_{j=1}^{n} \|x_j\|^2 = 1$. *If f is a continuous convex function defined on $[m, M]$,*
then

$$\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \tag{1.23}$$

$$\leq \frac{1}{M-m} \left[f(M) \sum_{j=1}^{n} \langle (A_j - mI) x_j, x_j \rangle + f(m) \sum_{j=1}^{n} \langle (MI - A_j) x_j, x_j \rangle \right].$$

Theorem 37 (Mićić-Seo-Takahasi-Tominaga, 1999, [4]) *Let A_j be self-adjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $m < M$ and $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. If f is a strictly convex function twice differentiable on $[m, M]$, then for any positive real number α we have*

$$\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \leq \alpha f \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) + \beta, \tag{1.24}$$

where

$$\beta = \mu_f t_0 + \nu_f - \alpha f(t_0),$$
$$\mu_f = \frac{f(M) - f(m)}{M - m}, \quad \nu_f = \frac{M f(m) - m f(M)}{M - m}$$

and

$$t_0 = \begin{cases} f'^{-1} \left(\frac{\mu_f}{\alpha} \right) & \text{if } m < f'^{-1} \left(\frac{\mu_f}{\alpha} \right) < M \\ M & \text{if } M \leq f'^{-1} \left(\frac{\mu_f}{\alpha} \right) \\ m & \text{if } f'^{-1} \left(\frac{\mu_f}{\alpha} \right) \leq m. \end{cases}$$

The case of equality was also analyzed, see [1, p. 61] but will be not stated in here.

1.6.3 Operator Monotone and Operator Convex Functions

We say that a real valued continuous function f defined on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e. if A and B are bounded selfadjoint operators with $A \leq B$ and $Sp(A), Sp(A) \subset I$, then $f(A) \leq f(B)$. The function is said to be *operator convex (operator concave)* if for any A, B bounded selfadjoint operators with $Sp(A), Sp(A) \subset I$, we have

$$f[(1 - \lambda) A + \lambda B] \leq (\geq) (1 - \lambda) f(A) + \lambda f(B) \tag{1.25}$$

for any $\lambda \in [0, 1]$.

Example 38 *The following examples are well know in the literature and can be found for instance in [1, p. 7-p. 9] where simple proofs were also provided.*

1. *The affine function $f(t) = \alpha + \beta t$ is operator monotone on every interval for all $\alpha \in \mathbb{R}$ and $\beta \geq 0$. It is operator convex for all $\alpha, \beta \in \mathbb{R}$;*

2. *If f, g are operator monotone, and if $\alpha, \beta \geq 0$ then the linear combination $\alpha f + \beta g$ is also operator monotone. If the functions f_n are operator monotone and $f_n(t) \to f(t)$ as $n \to \infty$, then f is also operator monotone;*

3. *The function $f(t) = t^2$ is operator convex on every interval, however it is not operator monotone on $[0, \infty)$ even though it is monotonic nondecreasing on this interval;*

4. *The function $f(t) = t^3$ is not operator convex on $[0, \infty)$ even though it is a convex function on this interval;*

5. *The function $f(t) = \frac{1}{t}$ is operator convex on $(0, \infty)$ and $f(t) = -\frac{1}{t}$ is operator monotone on $(0, \infty)$;*

6. *The function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$;*

7. *The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$;*

8. *The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone on any interval of $\mathbb{R}$.*

The following monotonicity property for the function $f(t) = t^r$ with $r \in [0, 1]$ is well known in the literature as the *Löwner-Heinz inequality* and was established essentially in 1934:

Theorem 39 (Löwner-Heinz Inequality) *Let A and B be positive operators on a Hilbert space H. If $A \geq B \geq 0$, then $A^r \geq B^r$ for all $r \in [0, 1]$.*

The following characterization of operator convexity holds, see [1, p. 10]

Theorem 40 (Jensen's Operator Inequality) *Let H and K be Hilbert spaces. Let f be a real valued continuous function on an interval J. Let A and A_j be selfadjoint operators on H with spectra contained in J, for each $j = 1, 2, ..., k$. Then the following conditions are mutually equivalent:*

(i) *f is operator convex on J;*

(ii) *$f(C^*AC) \leq C^* f(A) C$ for every selfadjoint operator $A : H \to H$ and isometry $C : K \to H, i.e., C^*C = 1_K$;*

(iii) $f\left(C^{*}AC\right) \leq C^{*}f\left(A\right)C$ for every selfadjoint operator $A : H \to H$ and isometry $C : H \to H$;

(iv) $f\left(\sum_{j=1}^{k} C_{j}^{*}A_{j}C_{j}\right) \leq \sum_{j=1}^{k} C_{j}^{*}f\left(A_{j}\right)C_{j}$ for every selfadjoint operator $A_{j} : H \to H$ and bounded linear operators $C_{j} : K \to H$, with $\sum_{j=1}^{k} C_{j}^{*}C_{j} = 1_{K}$ $(j = 1, ..., k)$;

(v) $f\left(\sum_{j=1}^{k} C_{j}^{*}A_{j}C_{j}\right) \leq \sum_{j=1}^{k} C_{j}^{*}f\left(A_{j}\right)C_{j}$ for every selfadjoint operator $A_{j} : H \to H$ and bounded linear operators $C_{j} : H \to H$, with $\sum_{j=1}^{k} C_{j}^{*}C_{j} = 1_{H}$ $(j = 1, ..., k)$;

(vi) $f\left(\sum_{j=1}^{k} P_{j}A_{j}P_{j}\right) \leq \sum_{j=1}^{k} P_{j}f\left(A_{j}\right)P_{j}$ for every selfadjoint operator $A_{j} : H \to H$ and projection $P_{j} : H \to H$, with $\sum_{j=1}^{k} P_{j} = 1_{H}$ $(j = 1, ..., k)$.

The following well known result due to Hansen & Pedersen also holds:

Theorem 41 (Hansen-Pedersen-Jensen's Inequality) *Let J be an interval containing* 0 *and let* f *be a real valued continuous function defined on J. Let A and* A_{j} *be selfadjoint operators on H with spectra contained in J, for each* $j = 1, 2, ..., k$. *Then the following conditions are mutually equivalent:*

(i) f *is operator convex on J and* $f\left(0\right) \leq 0$;

(ii) $f\left(C^{*}AC\right) \leq C^{*}f\left(A\right)C$ *for every selfadjoint operator* $A : H \to H$ *and contraction* $C : H \to H, i.e., C^{*}C \leq 1_{H}$;

(iii) $f\left(\sum_{j=1}^{k} C_{j}^{*}A_{j}C_{j}\right) \leq \sum_{j=1}^{k} C_{j}^{*}f\left(A_{j}\right)C_{j}$ *for every selfadjoint operator* $A_{j} : H \to H$ *and bounded linear operators* $C_{j} : H \to H$, *with* $\sum_{j=1}^{k} C_{j}^{*}C_{j} \leq 1_{H}$ $(j = 1, ..., k)$;

(iv) $f\left(PAP\right) \leq Pf\left(A\right)P$ *for every selfadjoint operator* $A : H \to H$ *and projection P.*

The case of continuous and negative functions is as follows, [1, p. 13]:

Theorem 42 *Let f be continuous on* $[0, \infty)$. *If* $f\left(t\right) \leq 0$ *for all* $t \in [0, \infty)$, *then each of the conditions (i)-(vi) from Theorem 40 is equivalent with*

(vii) $-f$ *is an operator monotone function.*

Corollary 43 *Let f be a real valued continuous function mapping the positive half line* $[0, \infty)$ *into itself. Then f is operator monotone if and only if f is operator concave.*

The following result may be stated as well [1, p. 14]:

Theorem 44 *Let f be continuous on the interval $[0, r)$ with $r \leq \infty$. Then the following conditions are mutually equivalent:*

(i) *f is operator convex and $f(0) \leq 0$;*

(ii) *The function $t \mapsto \frac{f(t)}{t}$ is operator monotone on $(0, r)$.*

As a particular case of interest, we can state that [1, p. 15]:

Corollary 45 *Let f be continuous on $[0, \infty)$ and taking positive values. The function f is operator monotone if and only if the function $t \mapsto \frac{t}{f(t)}$ is operator monotone.*

Finally we recall the following result as well [1, p. 16]:

Theorem 46 *Let f be a real valued continuous function on the interval $J = [\alpha, \infty)$ and bounded below, i.e., there exists $m \in \mathbb{R}$ such that $m \leq f(t)$ for all $t \in J$. Then the following conditions are mutually equivalent:*

(i) *f is operator concave on J;*

(ii) *f is operator monotone on J.*

As a particular case of this result we note that, the function $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$.

1.7 Grüss' Type Inequalities

The following operator version of the Grüss inequality was obtained by Mond & Pečarić in [6]:

Theorem 47 (Mond-Pečarić, 1993, [6]) *Let C_j, $j \in \{1, \ldots, n\}$ be self-adjoint operators on the Hilbert space $(H, \langle ., . \rangle)$ and such that $m_j \cdot 1_H \leq C_j \leq M_j \cdot 1_H$ for $j \in \{1, \ldots, n\}$, where 1_H is the identity operator on H. Further, let $g_j, h_j : [m_j, M_j] \to \mathbb{R}$, $j \in \{1, \ldots, n\}$ be functions such that*

$$\varphi \cdot 1_H \leq g_j(C_j) \leq \Phi \cdot 1_H \quad and \quad \gamma \cdot 1_H \leq h_j(C_j) \leq \Gamma \cdot 1_H \quad (1.26)$$

for each $j \in \{1, \ldots, n\}$.

If $x_j \in H$, $j \in \{1, \ldots, n\}$ are such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

$$\left| \sum_{j=1}^{n} \langle g_j(C_j) h_j(C_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle g_j(C_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle h_j(C_j) x_j, x_j \rangle \right|$$

$$\leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma). \quad (1.27)$$

If $C_j, j \in \{1, \ldots, n\}$ are selfadjoint operators such that $Sp\,(C_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$ and if $g, h : [m, M] \longrightarrow \mathbb{R}$ are continuous then by the Mond-Pečarić inequality we deduce the following version of the Grüss inequality for operators

$$\left| \sum_{j=1}^{n} \langle g\,(C_j)\,h\,(C_j)\,x_j, x_j \rangle - \sum_{j=1}^{n} \langle g\,(C_j)\,x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle h\,(C_j)\,x_j, x_j \rangle \right| \quad (1.28)$$

$$\leq \frac{1}{4}\,(\Phi - \varphi)\,(\Gamma - \gamma)\,,$$

where $x_j \in H$, $j \in \{1, \ldots, n\}$ are such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$ and $\varphi = \min_{t \in [m,M]} g\,(t)$, $\Phi = \max_{t \in [m,M]} g\,(t)$, $\gamma = \min_{t \in [m,M]} h\,(t)$ and $\Gamma = \max_{t \in [m,M]} h\,(t)$.

In particular, if the selfadjoint operator C satisfy the condition $Sp\,(C) \subseteq [m, M]$ for some scalars $m < M$, then

$$|\langle g\,(C)\,h\,(C)\,x, x \rangle - \langle g\,(C)\,x, x \rangle \cdot \langle h\,(C)\,x, x \rangle| \leq \frac{1}{4}\,(\Phi - \varphi)\,(\Gamma - \gamma)\,, \quad (1.29)$$

for any $x \in H$ with $\|x\| = 1$.

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2
Inequalities for Convex Functions

2.1 Introduction

Jensen's type inequalities in their various settings ranging from discrete to continuous case play an important role in different branches of Modern Mathematics. A simple search in the *MathSciNet* database of the American Mathematical Society with the key words "jensen" and "inequality" in the title reveals more than 300 items intimately devoted to this famous result. However, the number of papers where this inequality is applied is a lot larger and far more difficult to find. It can be a good project in itself for someone to write a monograph devoted to Jensen's inequality in its different forms and its applications across Mathematics.

In the introductory chapter we have recalled a number of Jensen's type inequalities for convex and operator convex functions of selfadjoint operators in Hilbert spaces. In this chapter we present some recent results obtained by the author that deal with different aspects of this well research inequality than those recently reported in the book [20]. They include but are not restricted to the operator version of the Dragomir-Ionescu inequality, Slater's type inequalities for operators and its inverses, Jensen's inequality for twice differentiable functions whose second derivatives satisfy some upper and lower bounds conditions, Jensen's type inequalities for log-convex functions and for differentiable log-convex functions and their applications to Ky Fan's inequality.

Finally, some Hermite-Hadamard's type inequalities for convex functions and Hermite-Hadamard's type inequalities for operator convex functions are presented as well.

All the above results are exemplified for some classes of elementary functions of interest such as the power function and the logarithmic function.

2.2   Reverses of the Jensen Inequality

2.2.1   An Operator Version of the Dragomir-Ionescu Inequality

The following result holds:

Theorem 48 (Dragomir, 2008, [9]) *Let I be an interval and $f : I \rightarrow \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp\,(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$(0 \leq) \langle f\,(A)\,x, x\rangle - f\,(\langle Ax, x\rangle) \leq \langle f'\,(A)\,Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'\,(A)\,x, x\rangle \quad (2.1)$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f is convex and differentiable, we have that

$$f\,(t) - f\,(s) \leq f'\,(t) \cdot (t - s)$$

for any $t, s \in [m, M]$.

Now, if we chose in this inequality $s = \langle Ax, x\rangle \in [m, M]$ for any $x \in H$ with $\|x\| = 1$ since $Sp\,(A) \subseteq [m, M]$, then we have

$$f\,(t) - f\,(\langle Ax, x\rangle) \leq f'\,(t) \cdot (t - \langle Ax, x\rangle) \quad (2.2)$$

for any $t \in [m, M]$ any $x \in H$ with $\|x\| = 1$.

If we fix $x \in H$ with $\|x\| = 1$ in (2.2) and apply the property (P) then we get

$$\langle [f\,(A) - f\,(\langle Ax, x\rangle)\,1_H]\,x, x\rangle \leq \langle f'\,(A) \cdot (A - \langle Ax, x\rangle\,1_H)\,x, x\rangle$$

for each $x \in H$ with $\|x\| = 1$, which is clearly equivalent to the desired inequality (2.1). ∎

Corollary 49 (Dragomir, 2008, [9]) *Assume that f is as in the Theorem 48. If A_j are selfadjoint operators with $Sp\,(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in$*

$\{1, \ldots, n\}$ and $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

$$(0 \le) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) \tag{2.3}$$

$$\le \sum_{j=1}^{n} \langle f'(A_j) A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle.$$

Proof. As in [20, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$,

$$\left\langle f\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle, \qquad \left\langle \widetilde{A}\widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle$$

and so on.

Applying Theorem 48 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result (2.3). ∎

Corollary 50 (Dragomir, 2008, [9]) *Assume that f is as in the Theorem 48. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, \ldots, n\}$ and $p_j \ge 0$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$(0 \le) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right) \tag{2.4}$$

$$\le \left\langle \sum_{j=1}^{n} p_j f'(A_j) A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle.$$

for each $x \in H$ with $\|x\| = 1$.

Remark 51 *The inequality (2.4), in the scalar case, namely*

$$(0 \le) \sum_{j=1}^{n} p_j f(x_j) - f\left(\sum_{j=1}^{n} p_j x_j\right) \tag{2.5}$$

$$\le \sum_{j=1}^{n} p_j f'(x_j) x_j - \sum_{j=1}^{n} p_j x_j \cdot \sum_{j=1}^{n} p_j f'(x_j),$$

where $x_j \in \mathring{I}$, $j \in \{1, \ldots, n\}$, has been obtained by the first time in 1994 by Dragomir & Ionescu, see [17].

The following particular cases are of interest:

Example 52 *a*. *Let A be a positive definite operator on the Hilbert space H. Then we have the following inequality:*

$$(0 \leq) \ln\left(\langle Ax, x \rangle\right) - \langle \ln\left(A\right)x, x \rangle \leq \langle Ax, x \rangle \cdot \langle A^{-1}x, x \rangle - 1, \qquad (2.6)$$

for each $x \in H$ with $\|x\| = 1$.
    b. *If A is a selfadjoint operator on H, then we have the inequality:*

$$(0 \leq) \langle \exp\left(A\right)x, x \rangle - \exp\left(\langle Ax, x \rangle\right) \qquad (2.7)$$
$$\leq \langle A \exp\left(A\right)x, x \rangle - \langle Ax, x \rangle \cdot \langle \exp\left(A\right)x, x \rangle,$$

for each $x \in H$ with $\|x\| = 1$.
    c. *If $p \geq 1$ and A is a positive operator on H, then*

$$(0 \leq) \langle A^p x, x \rangle - \langle Ax, x \rangle^p \leq p\left[\langle A^p x, x \rangle - \langle Ax, x \rangle \cdot \langle A^{p-1}x, x \rangle\right], \qquad (2.8)$$

for each $x \in H$ with $\|x\| = 1$. If A is positive definite, then the inequality (2.8) also holds for $p < 0$.
    If $0 < p < 1$ and A is a positive definite operator then the reverse inequality also holds

$$\langle A^p x, x \rangle - \langle Ax, x \rangle^p \geq p\left[\langle A^p x, x \rangle - \langle Ax, x \rangle \cdot \langle A^{p-1}x, x \rangle\right] \geq 0, \qquad (2.9)$$

for each $x \in H$ with $\|x\| = 1$.

Similar results can be stated for sequences of operators, however the details are omitted.

2.2.2   Further Reverses

In applications would be perhaps more useful to find upper bounds for the quantity

$$\langle f\left(A\right)x, x \rangle - f\left(\langle Ax, x \rangle\right), \quad x \in H \quad \text{with} \quad \|x\| = 1,$$

that are in terms of the spectrum margins m, M and of the function f.
    The following result may be stated:

Theorem 53 (Dragomir, 2008, [9]) *Let I be an interval and $f : I \rightarrow \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert*

space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then

$$(0 \leq) \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \tag{2.10}$$

$$\leq \begin{cases} \frac{1}{2} \cdot (M - m) \left[\|f'(A) x\|^2 - \langle f'(A) x, x \rangle^2 \right]^{1/2} \\ \\ \frac{1}{2} \cdot (f'(M) - f'(m)) \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases}$$

$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m)),$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequality

$$(0 \leq) \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \tag{2.11}$$

$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m))$$

$$- \begin{cases} [\langle Mx - Ax, Ax - mx \rangle \langle f'(M) x - f'(A) x, f'(A) x - f'(m) x \rangle]^{\frac{1}{2}}, \\ \\ \left| \langle Ax, x \rangle - \frac{M+m}{2} \right| \left| \langle f'(A) x, x \rangle - \frac{f'(M) + f'(m)}{2} \right| \end{cases}$$

$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m)),$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if $m > 0$ and $f'(m) > 0$, then we also have

$$(0 \leq) \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \tag{2.12}$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(f'(M) - f'(m)\right)}{\sqrt{Mm f'(M) f'(m)}} \langle Ax, x \rangle \langle f'(A) x, x \rangle, \\ \\ \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{f'(M)} - \sqrt{f'(m)} \right) [\langle Ax, x \rangle \langle f'(A) x, x \rangle]^{\frac{1}{2}}, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. We use the following Grüss' type result we obtained in [6]:

Let A be a selfadjoint operator on the Hilbert space $(H; \langle ., . \rangle)$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If h and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} h(t)$ and $\Gamma := \max_{t \in [m,M]} h(t)$, then

$$|\langle h(A) g(A) x, x \rangle - \langle h(A) x, x \rangle \cdot \langle g(A) x, x \rangle| \tag{2.13}$$

$$\leq \frac{1}{2} \cdot (\Gamma - \gamma) \left[\|g(A) x\|^2 - \langle g(A) x, x \rangle^2 \right]^{1/2}$$

$$\left(\leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta) \right)$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$.

Therefore, we can state that

$$\langle Af'(A) x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A) x, x \rangle \tag{2.14}$$
$$\leq \frac{1}{2} \cdot (M - m) \left[\|f'(A) x\|^2 - \langle f'(A) x, x \rangle^2 \right]^{1/2}$$
$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m))$$

and

$$\langle Af'(A) x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A) x, x \rangle \tag{2.15}$$
$$\leq \frac{1}{2} \cdot (f'(M) - f'(m)) \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2}$$
$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m))$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (2.10).

On making use of the inequality obtained in [7]:

$$|\langle h(A) g(A) x, x \rangle - \langle h(A) x, x \rangle \langle g(A) x, x \rangle| \tag{2.16}$$
$$\leq \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta)$$
$$- \begin{cases} [\langle \Gamma x - h(A) x, f(A) x - \gamma x \rangle \langle \Delta x - g(A) x, g(A) x - \delta x \rangle]^{\frac{1}{2}}, \\ \left| \langle h(A) x, x \rangle - \frac{\Gamma + \gamma}{2} \right| \left| \langle g(A) x, x \rangle - \frac{\Delta + \delta}{2} \right|, \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, we can state that

$$\langle Af'(A) x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A) x, x \rangle$$
$$\leq \frac{1}{4} (M - m)(f'(M) - f'(m))$$
$$- \begin{cases} [\langle Mx - Ax, Ax - mx \rangle \langle f'(M) x - f'(A) x, f'(A) x - f'(m) x \rangle]^{\frac{1}{2}}, \\ \left| \langle Ax, x \rangle - \frac{M + m}{2} \right| \left| \langle f'(A) x, x \rangle - \frac{f'(M) + f'(m)}{2} \right|. \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (2.11).

Further, in order to prove the third inequality, we make use of the following result of Grüss type obtained in [7]:

If γ and δ are positive, then

$$|\langle h(A) g(A) x, x \rangle - \langle h(A) x, x \rangle \langle g(A) x, x \rangle| \tag{2.17}$$
$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \langle h(A) x, x \rangle \langle g(A) x, x \rangle, \\ \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right) [\langle h(A) x, x \rangle \langle g(A) x, x \rangle]^{\frac{1}{2}}. \end{cases}$$

for each $x \in H$ with $\|x\| = 1$.

Now, on making use of (2.17) we can state that

$$\langle Af'(A) x, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A) x, x \rangle$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \langle Ax, x \rangle \langle f'(A) x, x \rangle, \\ \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right) [\langle Ax, x \rangle \langle f'(A) x, x \rangle]^{\frac{1}{2}}. \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (2.12). ∎

Corollary 54 (Dragomir, 2008, [9]) *Assume that f is as in the Theorem 53. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}, j \in \{1, \ldots, n\}$, then*

$$(0 \leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) \tag{2.18}$$

$$\leq \begin{cases} \frac{1}{2} \cdot (M - m) \left[\sum_{j=1}^{n} \|f'(A_j) x_j\|^2 - \left(\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle\right)^2\right]^{1/2}, \\ \\ \frac{1}{2} \cdot (f'(M) - f'(m)) \left[\sum_{j=1}^{n} \|A_j x_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^2\right]^{1/2}, \end{cases}$$

$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m)),$$

for any $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We also have the inequality

$$(0 \leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) \tag{2.19}$$

$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m))$$

$$- \begin{cases} \left[\sum_{j=1}^{n} \langle Mx_j - A_j x, A_j x_j - mx_j \rangle\right]^{\frac{1}{2}} \\ \times \left[\sum_{j=1}^{n} \langle f'(M) x_j - f'(A_j) x_j, f'(A_j) x_j - f'(m) x_j \rangle\right]^{1/2}, \\ \\ \left|\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - \frac{M+m}{2}\right| \left|\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases}$$

$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m)),$$

for any $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.
  Moreover, if $m > 0$ and $f'(m) > 0$, then we also have

$$(0 \leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \tag{2.20}$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle, \\[2ex] \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{f'(M)} - \sqrt{f'(m)} \right) \\[1ex] \times \left[\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle \right]^{\frac{1}{2}}, \end{cases}$$

for any $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

The following corollary also holds:

Corollary 55 (Dragomir, 2008, [9]) *Assume that f is as in the Theorem 48. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}, j \in \{1, \ldots, n\}$ and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$(0 \leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \tag{2.21}$$

$$\leq \begin{cases} \frac{1}{2} \cdot (M - m) \left[\sum_{j=1}^{n} p_j \|f'(A_j) x\|^2 - \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle^2 \right]^{1/2}, \\[3ex] \frac{1}{2} \cdot (f'(M) - f'(m)) \left[\sum_{j=1}^{n} p_j \|A_j x\|^2 - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^2 \right]^{1/2}, \end{cases}$$

$$\leq \frac{1}{4} (M - m)(f'(M) - f'(m)),$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequality

$$(0 \leq) \left\langle \sum_{j=1}^{n} p_j f\left(A_j\right) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right) \qquad (2.22)$$

$$\leq \frac{1}{4}\left(M - m\right)\left(f'\left(M\right) - f'\left(m\right)\right)$$

$$- \begin{cases} \left[\sum_{j=1}^{n} p_j \left\langle Mx - A_j x, A_j x - mx \right\rangle\right]^{\frac{1}{2}} \\ \times \left[\sum_{j=1}^{n} p_j \left\langle f'\left(M\right) x - f'\left(A_j\right) x, f'\left(A_j\right) x - f'\left(m\right) x \right\rangle\right]^{1/2} , \\ \left|\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - \frac{M+m}{2}\right| \left|\left\langle \sum_{j=1}^{n} p_j f'\left(A_j\right) x, x \right\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases}$$

$$\leq \frac{1}{4}\left(M - m\right)\left(f'\left(M\right) - f'\left(m\right)\right),$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if $m > 0$ and $f'\left(m\right) > 0$, then we also have

$$(0 \leq) \left\langle \sum_{j=1}^{n} p_j f\left(A_j\right) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right) \qquad (2.23)$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'\left(A_j\right) x, x \right\rangle, \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'\left(M\right)} - \sqrt{f'\left(m\right)}\right) \\ \times \left[\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'\left(A_j\right) x, x \right\rangle\right]^{\frac{1}{2}}, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Remark 56 *Some of the inequalities in Corollary 55 can be used to produce reverse norm inequalities for the sum of positive operators in the case when the convex function f is nonnegative and monotonic nondecreasing on $[0, M]$.*

For instance, if we use the inequality (2.21), then we have

$$(0 \ \leq \ ) \left\|\sum_{j=1}^{n} p_j f\left(A_j\right)\right\| - f\left(\left\|\sum_{j=1}^{n} p_j A_j\right\|\right) \qquad (2.24)$$

$$\leq \ \frac{1}{4}\left(M - m\right)\left(f'\left(M\right) - f'\left(m\right)\right).$$

Moreover, if we use the inequality (2.23), then we obtain

$$
(0 \leq) \left\| \sum_{j=1}^{n} p_j f\left(A_j\right) \right\| - f\left(\left\| \sum_{j=1}^{n} p_j A_j \right\| \right) \tag{2.25}
$$

$$
\leq
\begin{cases}
\frac{1}{4} \cdot \frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \left\| \sum_{j=1}^{n} p_j A_j \right\| \left\| \sum_{j=1}^{n} p_j f'\left(A_j\right) \right\|, \\[4mm]
\left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'\left(M\right)}-\sqrt{f'\left(m\right)}\right) \left[\left\| \sum_{j=1}^{n} p_j A_j \right\| \left\| \sum_{j=1}^{n} p_j f'\left(A_j\right) \right\| \right]^{\frac{1}{2}}.
\end{cases}
$$

2.2.3   Some Particular Inequalities of Interest

1. Consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f\left(x\right) = -\ln x$. On utilising the inequality (2.10), then for any positive definite operator A on the Hilbert space H, we have the inequality

$$
(0 \leq) \ln\left(\langle Ax, x\rangle\right) - \langle \ln\left(A\right)x, x\rangle \tag{2.26}
$$

$$
\leq
\begin{cases}
\frac{1}{2} \cdot (M-m) \left[\left\|A^{-1}x\right\|^2 - \left\langle A^{-1}x, x\right\rangle^2 \right]^{1/2} \\[3mm]
\frac{1}{2} \cdot \frac{M-m}{mM} \left[\left\|Ax\right\|^2 - \left\langle Ax, x\right\rangle^2 \right]^{1/2}
\end{cases}
$$

$$
\left(\leq \frac{1}{4} \cdot \frac{(M-m)^2}{mM} \right)
$$

for any $x \in H$ with $\|x\| = 1$.

However, if we use the inequality (2.11), then we have the following result as well

$$
(0 \leq) \ln\left(\langle Ax, x\rangle\right) - \langle \ln\left(A\right)x, x\rangle \tag{2.27}
$$

$$
\leq \frac{1}{4} \cdot \frac{(M-m)^2}{mM}
$$

$$
-
\begin{cases}
\left[\langle Mx - Ax, Ax - mx\rangle \left\langle M^{-1}x - A^{-1}x, A^{-1}x - m^{-1}x\right\rangle \right]^{\frac{1}{2}}, \\[3mm]
\left| \langle Ax, x\rangle - \frac{M+m}{2} \right| \left| \left\langle A^{-1}x, x\right\rangle - \frac{M+m}{2mM} \right|
\end{cases}
$$

$$
\left(\leq \frac{1}{4} \cdot \frac{(M-m)^2}{mM} \right)
$$

for any $x \in H$ with $\|x\| = 1$.

2. Now consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f\left(x\right) = x \ln x$. On utilising the inequality (2.10), then for any positive definite operator A on

the Hilbert space H, we have the inequality

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \tag{2.28}$$

$$\leq \begin{cases} \frac{1}{2} \cdot (M - m) \left[\| \ln (eA) x \|^2 - \langle \ln (eA) x, x \rangle^2 \right]^{1/2} \\[3mm] \ln \sqrt{\frac{M}{m}} \cdot \left[\| Ax \|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases}$$

$$\left(\leq \frac{1}{2} (M - m) \ln \sqrt{\frac{M}{m}} \right)$$

for any $x \in H$ with $\|x\| = 1$.

If we now apply the inequality (2.11), then we have the following result as well

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \tag{2.29}$$

$$\leq \frac{1}{2} (M - m) \ln \sqrt{\frac{M}{m}}$$

$$- \begin{cases} [\langle Mx - Ax, Ax - mx \rangle \langle \ln (M) x - \ln (A) x, \ln (A) x - \ln (m) x \rangle]^{\frac{1}{2}}, \\[3mm] \left| \langle Ax, x \rangle - \frac{M+m}{2} \right| \left| \langle \ln (A) x, x \rangle - \ln \sqrt{mM} \right| \end{cases}$$

$$\left(\leq \frac{1}{2} (M - m) \ln \sqrt{\frac{M}{m}} \right)$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if we assume that $m > e^{-1}$, then, by utilising the inequality (2.12) we can state the inequality

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \tag{2.30}$$

$$\leq \begin{cases} \frac{1}{2} \cdot \frac{(M-m) \ln \sqrt{\frac{M}{m}}}{\sqrt{Mm \ln(eM) \ln(em)}} \langle Ax, x \rangle \langle \ln (eA) x, x \rangle, \\[3mm] \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{\ln (eM)} - \sqrt{\ln (em)} \right) [\langle Ax, x \rangle \langle \ln (eA) x, x \rangle]^{\frac{1}{2}}, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

3. Consider now the following convex function $f : \mathbb{R} \to (0, \infty)$, $f(x) = \exp(\alpha x)$ with $\alpha > 0$. If we apply the inequalities (2.10), (2.11) and (2.12) for $f(x) = \exp(\alpha x)$ and for a selfadjoint operator A, then we get the

following results

$$(0 \leq) \langle \exp(\alpha A) x, x \rangle - \exp(\alpha \langle Ax, x \rangle) \qquad (2.31)$$

$$\leq \begin{cases} \frac{1}{2} \cdot \alpha (M - m) \left[\|\exp(\alpha A) x\|^2 - \langle \exp(\alpha A) x, x \rangle^2 \right]^{1/2} \\[2mm] \frac{1}{2} \cdot \alpha (\exp(\alpha M) - \exp(\alpha m)) \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases}$$

$$\left(\leq \frac{1}{4} \alpha (M - m)(\exp(\alpha M) - \exp(\alpha m)) \right),$$

and

$$(0 \leq) \langle \exp(\alpha A) x, x \rangle - \exp(\alpha \langle Ax, x \rangle) \qquad (2.32)$$

$$\leq \frac{1}{4} \alpha (M - m)(\exp(\alpha M) - \exp(\alpha m))$$

$$- \alpha \begin{cases} [\langle Mx - Ax, Ax - mx \rangle]^{1/2} \\ \quad \times [\langle \exp(\alpha M) x - \exp(\alpha A) x, \exp(\alpha A) x - \exp(\alpha m) x \rangle]^{\frac{1}{2}}, \\[2mm] \left| \langle Ax, x \rangle - \frac{M+m}{2} \right| \left| \langle \exp(\alpha A) x, x \rangle - \frac{\exp(\alpha M) + \exp(\alpha m)}{2} \right| \end{cases}$$

$$\left(\leq \frac{1}{4} \alpha (M - m)(\exp(\alpha M) - \exp(\alpha m)) \right)$$

and

$$(0 \leq) \langle \exp(\alpha A) x, x \rangle - \exp(\alpha \langle Ax, x \rangle) \qquad (2.33)$$

$$\leq \alpha \times \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(\exp(\alpha M) - \exp(\alpha m))}{\sqrt{Mm} \exp\left[\frac{\alpha(M+m)}{2}\right]} \langle Ax, x \rangle \langle \exp(\alpha A) x, x \rangle, \\[3mm] \left(\sqrt{M} - \sqrt{m} \right) \left(\exp\left(\frac{\alpha M}{2}\right) - \exp\left(\frac{\alpha m}{2}\right) \right) \\ \quad \times [\langle Ax, x \rangle \langle \exp(\alpha A) x, x \rangle]^{\frac{1}{2}} \end{cases}$$

for any $x \in H$ with $\|x\| = 1$, respectively.

Now, consider the convex function $f : \mathbb{R} \to (0, \infty)$, $f(x) = \exp(-\beta x)$ with $\beta > 0$. If we apply the inequalities (2.10) and (2.11) for $f(x) = \exp(-\beta x)$ and for a selfadjoint operator A, then we get the following results

$$(0 \leq) \langle \exp(-\beta A) x, x \rangle - \exp(-\beta \langle Ax, x \rangle) \qquad (2.34)$$

$$\leq \beta \times \begin{cases} \frac{1}{2} \cdot (M - m) \left[\|\exp(-\beta A) x\|^2 - \langle \exp(-\beta A) x, x \rangle^2 \right]^{1/2} \\[2mm] \frac{1}{2} \cdot (\exp(-\beta m) - \exp(-\beta M)) \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases}$$

$$\left(\leq \frac{1}{4} \beta (M - m)(\exp(-\beta m) - \exp(-\beta M)) \right)$$

and

$$(0 \leq) \langle \exp(-\beta A) x, x \rangle - \exp(-\beta \langle Ax, x \rangle) \tag{2.35}$$

$$\leq \frac{1}{4} \beta (M - m) (\exp(-\beta m) - \exp(-\beta M))$$

$$- \beta \begin{cases} [\langle Mx - Ax, Ax - mx \rangle]^{1/2} \\ \times [\langle \exp(-\beta M) x - \exp(-\beta A) x, \exp(-\beta A) x - \exp(-\beta m) x \rangle]^{\frac{1}{2}}, \\ \left| \langle Ax, x \rangle - \frac{M+m}{2} \right| \left| \langle \exp(-\beta A) x, x \rangle - \frac{\exp(-\beta M) + \exp(-\beta m)}{2} \right| \end{cases}$$

$$\left(\leq \frac{1}{4} \beta (M - m) (\exp(-\beta m) - \exp(-\beta M)) \right)$$

for any $x \in H$ with $\|x\| = 1$, respectively.

4. Finally, if we consider the convex function $f : [0, \infty) \to [0, \infty)$, $f(x) = x^p$ with $p \geq 1$, then on applying the inequalities (2.10) and (2.11) for the positive operator A we have the inequalities

$$(0 \leq) \langle A^p x, x \rangle - \langle Ax, x \rangle^p \tag{2.36}$$

$$\leq p \times \begin{cases} \frac{1}{2} \cdot (M - m) \left[\left\| A^{p-1} x \right\|^2 - \langle A^{p-1} x, x \rangle^2 \right]^{1/2} \\ \frac{1}{2} \cdot (M^{p-1} - m^{p-1}) \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases}$$

$$\left(\leq \frac{1}{4} p (M - m) (M^{p-1} - m^{p-1}) \right)$$

and

$$(0 \leq) \langle A^p x, x \rangle - \langle Ax, x \rangle^p \tag{2.37}$$

$$\leq \frac{1}{4} p (M - m) (M^{p-1} - m^{p-1})$$

$$- p \begin{cases} [\langle Mx - Ax, Ax - mx \rangle \langle M^{p-1} x - A^{p-1} x, A^{p-1} x - m^{p-1} x \rangle]^{\frac{1}{2}}, \\ \left| \langle Ax, x \rangle - \frac{M+m}{2} \right| \left| \langle A^{p-1} x, x \rangle - \frac{M^{p-1} + m^{p-1}}{2} \right| \end{cases}$$

$$\left(\leq \frac{1}{4} p (M - m) (M^{p-1} - m^{p-1}) \right)$$

for any $x \in H$ with $\|x\| = 1$, respectively.

If the operator A is positive definite $(m > 0)$ then, by utilising the inequality (2.12), we have

$$(0 \leq) \langle A^p x, x \rangle - \langle Ax, x \rangle^p \tag{2.38}$$

$$\leq p \times \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(M^{p-1}-m^{p-1})}{M^{p/2}m^{p/2}} \langle Ax, x \rangle \langle A^{p-1}x, x \rangle, \\ \\ \left(\sqrt{M} - \sqrt{m}\right)\left(M^{(p-1)/2} - m^{(p-1)/2}\right)\left[\langle Ax, x \rangle \langle A^{p-1}x, x \rangle\right]^{\frac{1}{2}}, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Now, if we consider the convex function $f : [0, \infty) \to [0, \infty)$, $f(x) = -x^p$ with $p \in (0, 1)$, then from the inequalities (2.10) and (2.11) and for the positive definite operator A we have the inequalities

$$(0 \leq) \langle Ax, x \rangle^p - \langle A^p x, x \rangle \tag{2.39}$$

$$\leq p \times \begin{cases} \frac{1}{2} \cdot (M - m)\left[\left\|A^{p-1}x\right\|^2 - \langle A^{p-1}x, x \rangle^2\right]^{1/2} \\ \\ \frac{1}{2} \cdot \left(m^{p-1} - M^{p-1}\right)\left[\|Ax\|^2 - \langle Ax, x \rangle^2\right]^{1/2} \end{cases}$$

$$\left(\leq \frac{1}{4}p(M - m)\left(m^{p-1} - M^{p-1}\right)\right)$$

and

$$(0 \leq) \langle Ax, x \rangle^p - \langle A^p x, x \rangle \tag{2.40}$$

$$\leq \frac{1}{4}p(M - m)\left(m^{p-1} - M^{p-1}\right)$$

$$- p \begin{cases} \left[\langle Mx - Ax, Ax - mx \rangle \langle M^{p-1}x - A^{p-1}x, A^{p-1}x - m^{p-1}x \rangle\right]^{\frac{1}{2}}, \\ \\ \left|\langle Ax, x \rangle - \frac{M+m}{2}\right|\left|\langle A^{p-1}x, x \rangle - \frac{M^{p-1}+m^{p-1}}{2}\right| \end{cases}$$

$$\left(\leq \frac{1}{4}p(M - m)\left(m^{p-1} - M^{p-1}\right)\right)$$

for any $x \in H$ with $\|x\| = 1$, respectively.

Similar results may be stated for the convex function $f : (0, \infty) \to (0, \infty)$, $f(x) = x^p$ with $p < 0$. However the details are left to the interested reader.

2.3   Some Slater Type Inequalities

2.3.1   Slater Type Inequalities for Functions of Real Variables

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_- (x) \le f'_+ (x) \le f'_- (y) \le f'_+ (y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi \left(\mathring{I} \right) \subset \mathbb{R}$ and

$$f(x) \ge f(a) + (x - a) \varphi(a) \qquad \text{for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \le \varphi(x) \le f'_+(x) \qquad \text{for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following result is well known in the literature as *the Slater inequality:*

Theorem 57 (Slater, 1981, [37]) *If $f : I \to \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $x_i \in I, p_i \ge 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$ and $\sum_{i=1}^{n} p_i \varphi(x_i) \ne 0$, where $\varphi \in \partial f$, then*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \le f \left(\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \right). \tag{2.41}$$

As pointed out in [5, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \in I, \tag{2.42}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

2.3.2   Some Slater Type Inequalities for Operators

The following result holds:

Theorem 58 (Dragomir, 2008, [10]) *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose*

derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $Sp\,(A) \subseteq [m, M] \subset \mathring{I}$ and $f'\,(A)$ is a positive definite operator on H then

$$0 \leq f \left(\frac{\langle A f'\,(A)\,x, x \rangle}{\langle f'\,(A)\,x, x \rangle} \right) - \langle f\,(A)\,x, x \rangle \tag{2.43}$$
$$\leq f' \left(\frac{\langle A f'\,(A)\,x, x \rangle}{\langle f'\,(A)\,x, x \rangle} \right) \left[\frac{\langle A f'\,(A)\,x, x \rangle - \langle A x, x \rangle \, \langle f'\,(A)\,x, x \rangle}{\langle f'\,(A)\,x, x \rangle} \right],$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$f'\,(s) \cdot (t - s) \leq f\,(t) - f\,(s) \leq f'\,(t) \cdot (t - s) \tag{2.44}$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A, then for any $x \in H$ with $\|x\| = 1$ we have

$$\langle f'\,(A) \cdot (t \cdot 1_H - A)\,x, x \rangle \leq \langle [f\,(t) \cdot 1_H - f\,(A)]\,x, x \rangle \tag{2.45}$$
$$\leq \langle f'\,(t) \cdot (t \cdot 1_H - A)\,x, x \rangle$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

The inequality (2.45) is equivalent with

$$t \langle f'\,(A)\,x, x \rangle - \langle f'\,(A)\,Ax, x \rangle \ \leq \ f\,(t) - \langle f\,(A)\,x, x \rangle \tag{2.46}$$
$$\leq \ f'\,(t)\,t - f'\,(t) \langle Ax, x \rangle$$

for any $t \in [m, M]$ any $x \in H$ with $\|x\| = 1$.

Now, since A is selfadjoint with $mI \leq A \leq MI$ and $f'\,(A)$ is positive definite, then $mf'\,(A) \leq A f'\,(A) \leq M f'\,(A)$, i.e., $m \langle f'\,(A)\,x, x \rangle \leq \langle A f'\,(A)\,x, x \rangle \leq M \langle f'\,(A)\,x, x \rangle$ for any $x \in H$ with $\|x\| = 1$, which shows that

$$t_0 := \frac{\langle A f'\,(A)\,x, x \rangle}{\langle f'\,(A)\,x, x \rangle} \in [m, M] \quad \text{for any } x \in H \quad \text{with} \quad \|x\| = 1.$$

Finally, if we put $t = t_0$ in the equation (2.46), then we get the desired result (2.43). ∎

Remark 59 *It is important to observe that, the condition that $f'\,(A)$ is a positive definite operator on H can be replaced with the more general assumption that*

$$\frac{\langle A f'\,(A)\,x, x \rangle}{\langle f'\,(A)\,x, x \rangle} \in \mathring{I} \quad \text{for any } x \in H \quad \text{with} \quad \|x\| = 1, \tag{2.47}$$

which may be easily verified for particular convex functions f.

Remark 60 *Now, if the functions is concave on $\overset{\circ}{I}$ and the condition (2.47) holds, then we have the inequality*

$$0 \leq \langle f(A) x, x \rangle - f\left(\frac{\langle Af'(A) x, x \rangle}{\langle f'(A) x, x \rangle}\right) \tag{2.48}$$

$$\leq f'\left(\frac{\langle Af'(A) x, x \rangle}{\langle f'(A) x, x \rangle}\right)\left[\frac{\langle Ax, x \rangle \langle f'(A) x, x \rangle - \langle Af'(A) x, x \rangle}{\langle f'(A) x, x \rangle}\right],$$

for any $x \in H$ with $\|x\| = 1$.

The following examples are of interest:

Example 61 *If A is a positive definite operator on H, then*

$$(0 \leq) \langle \ln Ax, x \rangle - \ln\left(\langle A^{-1}x, x \rangle^{-1}\right) \leq \langle Ax, x \rangle \cdot \langle A^{-1}x, x \rangle - 1, \tag{2.49}$$

for any $x \in H$ with $\|x\| = 1$.

Indeed, we observe that if we consider the concave function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then

$$\frac{\langle Af'(A) x, x \rangle}{\langle f'(A) x, x \rangle} = \frac{1}{\langle A^{-1}x, x \rangle} \in (0, \infty), \quad \text{for any } x \in H \quad \text{with} \quad \|x\| = 1$$

and by the inequality (2.48) we deduce the desired result (2.49).

The following example concerning powers of operators is of interest as well:

Example 62 *If A is a positive definite operator on H, then for any $x \in H$ with $\|x\| = 1$ we have*

$$0 \leq \langle A^p x, x \rangle^{p-1} - \langle A^{p-1} x, x \rangle^p \tag{2.50}$$

$$\leq p \langle A^p x, x \rangle^{p-2} \left[\langle A^p x, x \rangle - \langle Ax, x \rangle \langle A^{p-1} x, x \rangle\right]$$

for $p \geq 1$,

$$0 \leq \langle A^{p-1} x, x \rangle^p - \langle A^p x, x \rangle^{p-1} \tag{2.51}$$

$$\leq p \langle A^p x, x \rangle^{p-2} \left[\langle Ax, x \rangle \langle A^{p-1} x, x \rangle - \langle A^p x, x \rangle\right]$$

for $0 < p < 1$, and

$$0 \leq \langle A^p x, x \rangle^{p-1} - \langle A^{p-1} x, x \rangle^p \tag{2.52}$$

$$\leq (-p) \langle A^p x, x \rangle^{p-2} \left[\langle Ax, x \rangle \langle A^{p-1} x, x \rangle - \langle A^p x, x \rangle\right]$$

for $p < 0$.

The proof follows from the inequalities (2.43) and (2.48) for the convex (concave) function $f(t) = t^p$, $p \in (-\infty, 0) \cup [1, \infty)$ ($p \in (0, 1)$) by performing the required calculation. The details are omitted.

2.3.3   Further Reverses

The following results that provide perhaps more useful upper bounds for
the nonnegative quantity

$$
f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \qquad \text{for } x \in H \qquad \text{with} \qquad \|x\| = 1,
$$

can be stated:

Theorem 63 (Dragomir, 2008, [10]) *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. Assume that A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive definite operator on H. If we define*

$$
B(f', A; x) := \frac{1}{\langle f'(A)x, x\rangle} \cdot f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)
$$

then

$$
(0 \leq) f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \tag{2.53}
$$

$$
\leq B(f', A; x) \times \begin{cases} \frac{1}{2} \cdot (M - m) \left[\|f'(A)x\|^2 - \langle f'(A)x, x\rangle^2\right]^{1/2} \\[2mm] \frac{1}{2} \cdot (f'(M) - f'(m)) \left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases}
$$

$$
\leq \frac{1}{4}(M - m)(f'(M) - f'(m)) B(f', A; x)
$$

and

$$
(0 \leq) f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \tag{2.54}
$$

$$
\leq B(f', A; x) \times \left[\frac{1}{4}(M - m)(f'(M) - f'(m))\right.
$$

$$
- \begin{cases} [\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\[2mm] \left|\langle Ax, x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x, x\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \left.\right]
$$

$$
\leq \frac{1}{4}(M - m)(f'(M) - f'(m)) B(f', A; x),
$$

for any $x \in H$ with $\|x\| = 1$, respectively.

Moreover, if A is a positive definite operator, then

$$(0 \leq) f\left(\frac{\langle Af'(A) x, x\rangle}{\langle f'(A) x, x\rangle}\right) - \langle f(A) x, x\rangle \tag{2.55}$$

$$\leq B(f', A; x)$$

$$\times \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mm f'(M) f'(m)}} \langle Ax, x\rangle \langle f'(A) x, x\rangle, \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)[\langle Ax, x\rangle\langle f'(A) x, x\rangle]^{\frac{1}{2}}, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. We use the following Grüss' type result we obtained in [6]:

Let A be a selfadjoint operator on the Hilbert space $(H; \langle ., .\rangle)$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If h and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m, M]} h(t)$ and $\Gamma := \max_{t \in [m, M]} h(t)$, then

$$|\langle h(A) g(A) x, x\rangle - \langle h(A) x, x\rangle \cdot \langle g(A) x, x\rangle| \tag{2.56}$$

$$\leq \frac{1}{2} \cdot (\Gamma - \gamma) \left[\|g(A) x\|^2 - \langle g(A) x, x\rangle^2\right]^{1/2}$$

$$\left(\leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta)\right),$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t \in [m, M]} g(t)$ and $\Delta := \max_{t \in [m, M]} g(t)$.

Therefore, we can state that

$$\langle Af'(A) x, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A) x, x\rangle \tag{2.57}$$

$$\leq \frac{1}{2} \cdot (M - m) \left[\|f'(A) x\|^2 - \langle f'(A) x, x\rangle^2\right]^{1/2}$$

$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m))$$

and

$$\langle Af'(A) x, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A) x, x\rangle \tag{2.58}$$

$$\leq \frac{1}{2} \cdot (f'(M) - f'(m)) \left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2}$$

$$\leq \frac{1}{4} (M - m) (f'(M) - f'(m)),$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.43) provide the desired result (2.53).

On making use of the inequality obtained in [7]

$$|\langle h(A) g(A) x, x \rangle - \langle h(A) x, x \rangle \langle g(A) x, x \rangle| \tag{2.59}$$

$$\leq \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta)$$

$$- \begin{cases} [\langle \Gamma x - h(A) x, f(A) x - \gamma x \rangle \langle \Delta x - g(A) x, g(A) x - \delta x \rangle]^{\frac{1}{2}}, \\ \left| \langle h(A) x, x \rangle - \frac{\Gamma + \gamma}{2} \right| \left| \langle g(A) x, x \rangle - \frac{\Delta + \delta}{2} \right|, \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, we can state that

$$\langle A f'(A) x, x \rangle - \langle A x, x \rangle \cdot \langle f'(A) x, x \rangle$$

$$\leq \frac{1}{4}(M - m)(f'(M) - f'(m))$$

$$- \begin{cases} [\langle M x - A x, A x - m x \rangle \langle f'(M) x - f'(A) x, f'(A) x - f'(m) x \rangle]^{\frac{1}{2}}, \\ \left| \langle A x, x \rangle - \frac{M + m}{2} \right| \left| \langle f'(A) x, x \rangle - \frac{f'(M) + f'(m)}{2} \right|, \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.43) provide the desired result (2.54).

Further, in order to prove the third inequality, we make use of the following result of Grüss type obtained in [7]:

If γ and δ are positive, then

$$|\langle h(A) g(A) x, x \rangle - \langle h(A) x, x \rangle \langle g(A) x, x \rangle| \tag{2.60}$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \langle h(A) x, x \rangle \langle g(A) x, x \rangle, \\ \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right) [\langle h(A) x, x \rangle \langle g(A) x, x \rangle]^{\frac{1}{2}}, \end{cases}$$

for each $x \in H$ with $\|x\| = 1$.

Now, on making use of (2.60) we can state that

$$\langle A f'(A) x, x \rangle - \langle A x, x \rangle \cdot \langle f'(A) x, x \rangle \tag{2.61}$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(M - m)(f'(M) - f'(m))}{\sqrt{M m f'(M) f'(m)}} \langle A x, x \rangle \langle f'(A) x, x \rangle, \\ \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{f'(M)} - \sqrt{f'(m)} \right) [\langle A x, x \rangle \langle f'(A) x, x \rangle]^{\frac{1}{2}}, \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.43) provide the desired result (2.55). ∎

Remark 64 *We observe, from the first inequality in (2.55), that*

$$(1 \leq) \frac{\langle A f'(A) x, x \rangle}{\langle A x, x \rangle \langle f'(A) x, x \rangle} \leq \frac{1}{4} \cdot \frac{(M - m)(f'(M) - f'(m))}{\sqrt{M m f'(M) f'(m)}} + 1$$

which implies that

$$f' \left(\frac{\langle Af'(A) x, x \rangle}{\langle f'(A) x, x \rangle} \right) \leq f' \left(\left[\frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} + 1 \right] \langle Ax, x \rangle \right),$$

for each $x \in H$ with $\|x\| = 1$, since f' is monotonic nondecreasing and A is positive definite.

Now, the first inequality in (2.55) implies the following result

$$(0 \leq) f \left(\frac{\langle Af'(A) x, x \rangle}{\langle f'(A) x, x \rangle} \right) - \langle f(A) x, x \rangle \tag{2.62}$$

$$\leq \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}$$

$$\times f' \left(\left[\frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} + 1 \right] \langle Ax, x \rangle \right) \langle Ax, x \rangle,$$

for each $x \in H$ with $\|x\| = 1$.

From the second inequality in (2.55) we also have

$$(0 \leq) f \left(\frac{\langle Af'(A) x, x \rangle}{\langle f'(A) x, x \rangle} \right) - \langle f(A) x, x \rangle \tag{2.63}$$

$$\leq \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{f'(M)} - \sqrt{f'(m)} \right)$$

$$\times f' \left(\left[\frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} + 1 \right] \langle Ax, x \rangle \right) \left[\frac{\langle Ax, x \rangle}{\langle f'(A) x, x \rangle} \right]^{\frac{1}{2}},$$

for each $x \in H$ with $\|x\| = 1$.

Remark 65 *If the condition that $f'(A)$ is a positive definite operator on H from the Theorem 63 is replaced by the condition (2.47), then the inequalities (2.53) and (2.56) will still hold. Similar inequalities for concave functions can be stated. However, the details are not provided here.*

2.3.4   Multivariate Versions

The following result for sequences of operators can be stated.

Theorem 66 (Dragomir, 2008, [10]) *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If $A_j, j \in \{1, \ldots, n\}$ are selfadjoint operators on the Hilbert space H with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$ and*

$$\frac{\sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle} \in \mathring{I} \tag{2.64}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then

$$0 \le f \left(\frac{\sum_{j=1}^n \langle A_j f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^n \langle f'(A_j) x_j, x_j \rangle} \right) - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \qquad (2.65)$$

$$\le f' \left(\frac{\sum_{j=1}^n \langle A_j f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^n \langle f'(A_j) x_j, x_j \rangle} \right)$$

$$\times \left[\frac{\sum_{j=1}^n \langle A_j f'(A_j) x_j, x_j \rangle - \sum_{j=1}^n \langle A_j x_j, x_j \rangle \sum_{j=1}^n \langle f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^n \langle f'(A_j) x_j, x_j \rangle} \right],$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

Proof. Follows from Theorem 58. The details are omitted. ∎

The following particular case is of interest

Corollary 67 (Dragomir, 2008, [10]) *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If $A_j, j \in \{1, \ldots, n\}$ are selfadjoint operators on the Hilbert space H with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$ and for $p_j \ge 0$ with $\sum_{j=1}^n p_j = 1$ if we also assume that*

$$\frac{\langle \sum_{j=1}^n p_j A_j f'(A_j) x, x \rangle}{\langle \sum_{j=1}^n p_j f'(A_j) x, x \rangle} \in \mathring{I} \qquad (2.66)$$

for each $x \in H$ with $\|x\| = 1$, then

$$0 \le f \left(\frac{\langle \sum_{j=1}^n p_j A_j f'(A_j) x, x \rangle}{\langle \sum_{j=1}^n p_j f'(A_j) x, x \rangle} \right) - \left\langle \sum_{j=1}^n p_j f(A_j) x, x \right\rangle \qquad (2.67)$$

$$\le f' \left(\frac{\langle \sum_{j=1}^n p_j A_j f'(A_j) x, x \rangle}{\langle \sum_{j=1}^n p_j f'(A_j) x, x \rangle} \right)$$

$$\times \left[\frac{\langle \sum_{j=1}^n p_j A_j f'(A_j) x, x \rangle - \langle \sum_{j=1}^n p_j A_j x, x \rangle \langle \sum_{j=1}^n p_j f'(A_j) x, x \rangle}{\langle \sum_{j=1}^n p_j f'(A_j) x, x \rangle} \right],$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Follows from Theorem 66 on choosing $x_j = \sqrt{p_j} \cdot x, j \in \{1, \ldots, n\}$, where $p_j \ge 0, j \in \{1, \ldots, n\}, \sum_{j=1}^n p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ∎

The following examples are interesting in themselves:

Example 68 *If A_j, $j \in \{1, \ldots, n\}$ are positive definite operators on H, then*

$$(0 \leq) \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle - \ln \left[\left(\sum_{j=1}^{n} \langle A_j^{-1} x_j, x_j \rangle \right)^{-1} \right] \tag{2.68}$$

$$\leq \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^{-1} x_j, x_j \rangle - 1,$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, then we also have the inequality

$$(0 \leq) \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle - \ln \left[\left(\left\langle \sum_{j=1}^{n} p_j A_j^{-1} x, x \right\rangle \right)^{-1} \right] \tag{2.69}$$

$$\leq \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{-1} x, x \right\rangle - 1,$$

for each $x \in H$ with $\|x\| = 1$.

2.4   Other Inequalities for Convex Functions

2.4.1   Some Inequalities for Two Operators

The following result holds:

Theorem 69 (Dragomir, 2008, [11]) *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A and B are selfadjoint operators on the Hilbert space H with $Sp(A), Sp(B) \subseteq [m, M] \subset \mathring{I}$, then*

$$\langle f'(A) x, x \rangle \langle By, y \rangle - \langle f'(A) Ax, x \rangle \tag{2.70}$$
$$\leq \langle f(B) y, y \rangle - \langle f(A) x, x \rangle$$
$$\leq \langle f'(B) By, y \rangle - \langle Ax, x \rangle \langle f'(B) y, y \rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.
  In particular, we have

$$\langle f'(A) x, x \rangle \langle Ay, y \rangle - \langle f'(A) Ax, x \rangle \tag{2.71}$$
$$\leq \langle f(A) y, y \rangle - \langle f(A) x, x \rangle$$
$$\leq \langle f'(A) Ay, y \rangle - \langle Ax, x \rangle \langle f'(A) y, y \rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\langle f'(A) x, x \rangle \langle Bx, x \rangle - \langle f'(A) Ax, x \rangle \qquad (2.72)$$
$$\leq \langle f(B) x, x \rangle - \langle f(A) x, x \rangle$$
$$\leq \langle f'(B) Bx, x \rangle - \langle Ax, x \rangle \langle f'(B) x, x \rangle$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s) \qquad (2.73)$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A, then for any $x \in H$ with $\|x\| = 1$ we have

$$\langle f'(A) \cdot (t \cdot 1_H - A) x, x \rangle \leq \langle [f(t) \cdot 1_H - f(A)] x, x \rangle \qquad (2.74)$$
$$\leq \langle f'(t) \cdot (t \cdot 1_H - A) x, x \rangle$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

The inequality (2.74) is equivalent with

$$t \langle f'(A) x, x \rangle - \langle f'(A) Ax, x \rangle \quad \leq \quad f(t) - \langle f(A) x, x \rangle \qquad (2.75)$$
$$\leq \quad f'(t) t - f'(t) \langle Ax, x \rangle$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

If we fix $x \in H$ with $\|x\| = 1$ in (2.75) and apply the property (P) for the operator B, then we get

$$\langle [\langle f'(A) x, x \rangle B - \langle f'(A) Ax, x \rangle 1_H] y, y \rangle$$
$$\leq \langle [f(B) - \langle f(A) x, x \rangle 1_H] y, y \rangle$$
$$\leq \langle [f'(B) B - \langle Ax, x \rangle f'(B)] y, y \rangle$$

for each $y \in H$ with $\|y\| = 1$, which is clearly equivalent to the desired inequality (2.70). ∎

Remark 70 *If we fix $x \in H$ with $\|x\| = 1$ and choose $B = \langle Ax, x \rangle \cdot 1_H$, then we obtain from the first inequality in (2.70) the reverse of the Mond-Pečarić inequality obtained by the author in [9]. The second inequality will provide the Mond-Pečarić inequality for convex functions whose derivatives are continuous.*

The following corollary is of interest:

Corollary 71 *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. Also, suppose that A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$. If g is nonincreasing and continuous on $[m, M]$ and*

$$f'(A) [g(A) - A] \geq 0 \qquad (2.76)$$

in the operator order of $B(H)$, *then*

$$(f \circ g)(A) \geq f(A) \tag{2.77}$$

in the operator order of $B(H)$.

Proof. If we apply the first inequality from (2.72) for $B = g(A)$ we have

$$\langle f'(A)x, x \rangle \langle g(A)x, x \rangle - \langle f'(A)Ax, x \rangle \tag{2.78}$$
$$\leq \langle f(g(A))x, x \rangle - \langle f(A)x, x \rangle$$

any $x \in H$ with $\|x\| = 1$.

We use the following Čebyšev type inequality for functions of operators established by the author in [8]:

Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $h, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and *synchronous (asynchronous)* on $[m, M]$, then

$$\langle h(A)g(A)x, x \rangle \geq (\leq) \langle h(A)x, x \rangle \cdot \langle g(A)x, x \rangle \tag{2.79}$$

for any $x \in H$ with $\|x\| = 1$.

Now, since f' and g are continuous and are asynchronous on $[m, M]$, then by (2.79) we have the inequality

$$\langle f'(A)g(A)x, x \rangle \leq \langle f'(A)x, x \rangle \cdot \langle g(A)x, x \rangle \tag{2.80}$$

for any $x \in H$ with $\|x\| = 1$.

Subtracting from both sides of (2.80) the quantity $\langle f'(A)Ax, x \rangle$ and taking into account, by (2.76), that $\langle f'(A)[g(A) - A]x, x \rangle \geq 0$ for any $x \in H$ with $\|x\| = 1$, we then have

$$0 \leq \langle f'(A)[g(A) - A]x, x \rangle$$
$$= \langle f'(A)g(A)x, x \rangle - \langle f'(A)Ax, x \rangle$$
$$\leq \langle f'(A)x, x \rangle \cdot \langle g(A)x, x \rangle - \langle f'(A)Ax, x \rangle$$

which together with (2.78) will produce the desired result (2.77). ∎

We provide now some particular inequalities of interest that can be derived from Theorem 69:

Example 72 *a.* *Let* A, B *two positive definite operators on* H. *Then we have the inequalities*

$$1 - \langle A^{-1}x, x \rangle \langle By, y \rangle \leq \langle \ln Ax, x \rangle - \langle \ln By, y \rangle \tag{2.81}$$
$$\leq \langle Ax, x \rangle \langle B^{-1}y, y \rangle - 1$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

In particular, we have

$$1 - \langle A^{-1}x, x \rangle \langle Ay, y \rangle \leq \langle \ln Ax, x \rangle - \langle \ln Ay, y \rangle \qquad (2.82)$$
$$\leq \langle Ax, x \rangle \langle A^{-1}y, y \rangle - 1$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$1 - \langle A^{-1}x, x \rangle \langle Bx, x \rangle \leq \langle \ln Ax, x \rangle - \langle \ln Bx, x \rangle \qquad (2.83)$$
$$\leq \langle Ax, x \rangle \langle B^{-1}x, x \rangle - 1$$

for any $x \in H$ with $\|x\| = 1$.

b. *With the same assumption for A and B we have the inequalities*

$$\langle By, y \rangle - \langle Ax, x \rangle \leq \langle B \ln By, y \rangle - \langle \ln Ax, x \rangle \langle By, y \rangle \qquad (2.84)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$\langle Ay, y \rangle - \langle Ax, x \rangle \leq \langle A \ln Ay, y \rangle - \langle \ln Ax, x \rangle \langle Ay, y \rangle \qquad (2.85)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\langle Bx, x \rangle - \langle Ax, x \rangle \leq \langle B \ln Bx, x \rangle - \langle \ln Ax, x \rangle \langle Bx, x \rangle \qquad (2.86)$$

for any $x \in H$ with $\|x\| = 1$.

The proof of Example **a** follows from Theorem 69 for the convex function $f(x) = -\ln x$ while the proof of the second example follows by the same theorem applied for the convex function $f(x) = x \ln x$ and performing the required calculations. The details are omitted.

The following result may be stated as well:

Theorem 73 (Dragomir, 2008, [11]) *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A and B are selfadjoint operators on the Hilbert space H with $Sp(A), Sp(B) \subseteq [m, M] \subset \mathring{I}$, then*

$$f'(\langle Ax, x \rangle)(\langle By, y \rangle - \langle Ax, x \rangle) \qquad (2.87)$$
$$\leq \langle f(B)y, y \rangle - f(\langle Ax, x \rangle)$$
$$\leq \langle f'(B)By, y \rangle - \langle Ax, x \rangle \langle f'(B)y, y \rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$f'(\langle Ax, x \rangle)(\langle Ay, y \rangle - \langle Ax, x \rangle) \qquad (2.88)$$
$$\leq \langle f(A)y, y \rangle - f(\langle Ax, x \rangle)$$
$$\leq \langle f'(A)Ay, y \rangle - \langle Ax, x \rangle \langle f'(A)y, y \rangle$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$
\begin{aligned}
f'\left(\langle Ax, x\rangle\right)\left(\langle Bx, x\rangle - \langle Ax, x\rangle\right) & \qquad (2.89)\\
\leq \langle f\left(B\right) x, x\rangle - f\left(\langle Ax, x\rangle\right) &\\
\leq \langle f'\left(B\right) Bx, x\rangle - \langle Ax, x\rangle \langle f'\left(B\right) x, x\rangle &
\end{aligned}
$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$
f'\left(s\right) \cdot \left(t - s\right) \leq f\left(t\right) - f\left(s\right) \leq f'\left(t\right) \cdot \left(t - s\right) \qquad (2.90)
$$

for any $t, s \in [m, M]$.

If we choose $s = \langle Ax, x\rangle \in [m, M]$, with a fix $x \in H$ with $\|x\| = 1$, then we have

$$
f'\left(\langle Ax, x\rangle\right) \cdot \left(t - \langle Ax, x\rangle\right) \leq f\left(t\right) - f\left(\langle Ax, x\rangle\right) \leq f'\left(t\right) \cdot \left(t - \langle Ax, x\rangle\right) \quad (2.91)
$$

for any $t \in [m, M]$.

Now, if we apply the property (P) to the inequality (2.91) and the operator B, then we get

$$
\begin{aligned}
\langle f'\left(\langle Ax, x\rangle\right) \cdot \left(B - \langle Ax, x\rangle \cdot 1_H\right) y, y\rangle & \qquad (2.92)\\
\leq \langle \left[f\left(B\right) - f\left(\langle Ax, x\rangle\right) \cdot 1_H\right] y, y\rangle &\\
\leq \langle f'\left(B\right) \cdot \left(B - \langle Ax, x\rangle \cdot 1_H\right) y, y\rangle &
\end{aligned}
$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which is equivalent with the desired result (2.87). ∎

Remark 74 *We observe that if we choose $B = A$ in (2.89) or $y = x$ in (2.88) then we recapture the Mond-Pečarić inequality and its reverse from (2.1).*

The following particular case of interest follows from Theorem 73

Corollary 75 (Dragomir, 2008, [11]) *Assume that f, A and B are as in Theorem 73. If, either f is increasing on $[m, M]$ and $B \geq A$ in the operator order of $B\left(H\right)$ or f is decreasing and $B \leq A$, then we have the Jensen's type inequality*

$$
\langle f\left(B\right) x, x\rangle \geq f\left(\langle Ax, x\rangle\right) \qquad (2.93)
$$

for any $x \in H$ *with* $\|x\| = 1$.

The proof is obvious by the first inequality in (2.89) and the details are omitted.

We provide now some particular inequalities of interest that can be derived from Theorem 73:

Example 76 a. *Let A, B be two positive definite operators on H. Then we have the inequalities*

$$1 - \langle Ax, x\rangle^{-1} \langle By, y\rangle \le \ln\left(\langle Ax, x\rangle\right) - \langle \ln By, y\rangle \qquad (2.94)$$
$$\le \langle Ax, x\rangle \langle B^{-1}y, y\rangle - 1$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$1 - \langle Ax, x\rangle^{-1} \langle Ay, y\rangle \le \ln\left(\langle Ax, x\rangle\right) - \langle \ln Ay, y\rangle \qquad (2.95)$$
$$\le \langle Ax, x\rangle \langle A^{-1}y, y\rangle - 1$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$1 - \langle Ax, x\rangle^{-1} \langle Bx, x\rangle \le \ln\left(\langle Ax, x\rangle\right) - \langle \ln Bx, x\rangle \qquad (2.96)$$
$$\le \langle Ax, x\rangle \langle B^{-1}x, x\rangle - 1$$

for any $x \in H$ with $\|x\| = 1$.

b. *With the same assumption for A and B, we have the inequalities*

$$\langle By, y\rangle - \langle Ax, x\rangle \le \langle B\ln By, y\rangle - \langle By, y\rangle \ln\left(\langle Ax, x\rangle\right) \qquad (2.97)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$\langle Ay, y\rangle - \langle Ax, x\rangle \le \langle A\ln Ay, y\rangle - \langle Ay, y\rangle \ln\left(\langle Ax, x\rangle\right) \qquad (2.98)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\langle Bx, x\rangle - \langle Ax, x\rangle \le \langle B\ln Bx, x\rangle - \langle Bx, x\rangle \ln\left(\langle Ax, x\rangle\right) \qquad (2.99)$$

for any $x \in H$ with $\|x\| = 1$.

2.4.2   Inequalities for Two Sequences of Operators

The following result may be stated:

Theorem 77 (Dragomir, 2008, [11]) *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A_j and B_j are selfadjoint operators on the Hilbert space H with $Sp(A_j), Sp(B_j) \subseteq [m, M] \subset \mathring{I}$ for any $j \in \{1, \ldots, n\}$, then*

$$\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j\rangle \sum_{j=1}^{n} \langle B_j y_j, y_j\rangle - \sum_{j=1}^{n} \langle f'(A_j) A_j x_j, x_j\rangle \qquad (2.100)$$

$$\le \sum_{j=1}^{n} \langle f(B_j) y_j, y_j\rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle$$

$$\le \sum_{j=1}^{n} \langle f'(B_j) B_j y_j, y_j\rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j\rangle \sum_{j=1}^{n} \langle f'(B_j) y_j, y_j\rangle$$

for any $x_j, y_j \in H$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n \|y_j\|^2 = 1$.

In particular, we have

$$\sum_{j=1}^n \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle A_j y_j, y_j \rangle - \sum_{j=1}^n \langle f'(A_j) A_j x_j, x_j \rangle \qquad (2.101)$$

$$\leq \sum_{j=1}^n \langle f(A_j) y_j, y_j \rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle$$

$$\leq \sum_{j=1}^n \langle f'(A_j) A_j y_j, y_j \rangle - \sum_{j=1}^n \langle A_j x_j, x_j \rangle \sum_{j=1}^n \langle f'(A_j) y_j, y_j \rangle$$

for any $x_j, y_j \in H$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n \|y_j\|^2 = 1$ and

$$\sum_{j=1}^n \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle B_j x_j, x_j \rangle - \sum_{j=1}^n \langle f'(A_j) A_j x_j, x_j \rangle \qquad (2.102)$$

$$\leq \sum_{j=1}^n \langle f(B_j) x_j, x_j \rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle$$

$$\leq \sum_{j=1}^n \langle f'(B_j) B_j x_j, x_j \rangle - \sum_{j=1}^n \langle A_j x_j, x_j \rangle \sum_{j=1}^n \langle f'(B_j) x_j, x_j \rangle$$

for any $x_j \in H$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

Proof. Follows from Theorem 69 and the details are omitted. ∎

The following particular case may be of interest:

Corollary 78 (Dragomir, 2008, [11]) *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A_j and B_j are selfadjoint operators on the Hilbert space H with $Sp(A_j), Sp(B_j) \subseteq [m, M] \subset \mathring{I}$ for any $j \in \{1, \ldots, n\}$, then for any $p_j, q_j \geq 0$ with $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1$, we have the inequalities*

$$\left\langle \sum_{j=1}^n p_j f'(A_j) x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j f'(A_j) A_j x, x \right\rangle \qquad (2.103)$$

$$\leq \left\langle \sum_{j=1}^n q_j f(B_j) y, y \right\rangle - \left\langle \sum_{j=1}^n p_j f(A_j) x, x \right\rangle$$

$$\leq \left\langle \sum_{j=1}^n q_j f'(B_j) B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n q_j f'(B_j) y, y \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j A_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j f'(A_j) A_j x, x \right\rangle \quad (2.104)$$

$$\leq \left\langle \sum_{j=1}^{n} q_j f(A_j) y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle$$

$$\leq \left\langle \sum_{j=1}^{n} q_j f'(A_j) B_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j f'(A_j) y, y \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j B_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f'(A_j) A_j x, x \right\rangle \quad (2.105)$$

$$\leq \left\langle \sum_{j=1}^{n} p_j f(B_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle$$

$$\leq \left\langle \sum_{j=1}^{n} p_j f'(B_j) B_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(B_j) x, x \right\rangle$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Follows from Theorem 77 on choosing $x_j = \sqrt{p_j} \cdot x$, $y_j = \sqrt{q_j} \cdot y$, $j \in \{1, \ldots, n\}$, where $p_j, q_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$ and $x, y \in H$, with $\|x\| = \|y\| = 1$. The details are omitted. ∎

Example 79 a. *Let A_j, B_j, $j \in \{1, \ldots, n\}$, be two sequences of positive definite operators on H. Then we have the inequalities*

$$1 - \sum_{j=1}^{n} \left\langle A_j^{-1} x_j, x_j \right\rangle \sum_{j=1}^{n} \left\langle B_j y_j, y_j \right\rangle \quad (2.106)$$

$$\leq \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle \ln B_j y_j, y_j \right\rangle$$

$$\leq \sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \sum_{j=1}^{n} \left\langle B_j^{-1} y_j, y_j \right\rangle - 1$$

for any $x_j, y_j \in H$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

b. *With the same assumption for A_j and B_j we have the inequalities*

$$\sum_{j=1}^{n} \langle B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \tag{2.107}$$

$$\leq \sum_{j=1}^{n} \langle B_j \ln B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle B_j y_j, y_j \rangle$$

for any $x_j, y_j \in H$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

Finally, we have

Example 80   a. *Let A_j, B_j, $j \in \{1, \ldots, n\}$, be two sequences of positive definite operators on H. Then for any $p_j, q_j \geq 0$ with $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$, we have the inequalities*

$$1 - \left\langle \sum_{j=1}^{n} p_j A_j^{-1} x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j B_j y, y \right\rangle \tag{2.108}$$

$$\leq \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} q_j \ln B_j y, y \right\rangle$$

$$\leq \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j B_j^{-1} y, y \right\rangle - 1$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

b. *With the same assumption for A_j, B_j, p_j and q_j, we have the inequalities*

$$\left\langle \sum_{j=1}^{n} q_j B_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \tag{2.109}$$

$$\leq \left\langle \sum_{j=1}^{n} q_j B_j \ln B_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j B_j y, y \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Remark 81 *We observe that all the other inequalities for two operators obtained in Subsection 3.1 can be extended for two sequences of operators in a similar way. However, the details are left to the interested reader.*

2.5   Some Jensen Type Inequalities for Twice Differentiable Functions

2.5.1   Jensen's Inequality for Twice Differentiable Functions

The following result may be stated:

Theorem 82 (Dragomir, 2008, [12]) *Let A be a positive definite operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If f is a twice differentiable function on (m, M) and for $p \in (-\infty, 0) \cup (1, \infty)$ we have for some $\gamma < \Gamma$ that*

$$\gamma \le \frac{t^{2-p}}{p(p-1)} \cdot f''(t) \le \Gamma \quad \text{for any} \quad t \in (m, M), \tag{2.110}$$

then

$$\gamma \left(\langle A^p x, x \rangle - \langle Ax, x \rangle^p \right) \le \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \tag{2.111}$$
$$\le \Gamma \left(\langle A^p x, x \rangle - \langle Ax, x \rangle^p \right)$$

for each $x \in H$ with $\|x\| = 1$.
    If

$$\delta \le \frac{t^{2-p}}{p(1-p)} \cdot f''(t) \le \Delta \quad \text{for any} \quad t \in (m, M) \tag{2.112}$$

and for some $\delta < \Delta$, where $p \in (0, 1)$, then

$$\delta \left(\langle Ax, x \rangle^p - \langle A^p x, x \rangle \right) \le \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \tag{2.113}$$
$$\le \Delta \left(\langle Ax, x \rangle^p - \langle A^p x, x \rangle \right)$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Consider the function $g_{\gamma,p} : (m, M) \to \mathbb{R}$ given by $g_{\gamma,p}(t) = f(t) - \gamma t^p$ where $p \in (-\infty, 0) \cup (1, \infty)$. The function $g_{\gamma,p}$ is twice differentiable,

$$g''_{\gamma,p}(t) = f''(t) - \gamma p(p-1) t^{p-2}$$

for any $t \in (m, M)$ and by (2.110) we deduce that $g_{\gamma,p}$ is convex on (m, M). Now, applying the Mond & Pečarić inequality for $g_{\gamma,p}$ we have

$$0 \le \langle (f(A) - \gamma A^p) x, x \rangle - [f(\langle Ax, x \rangle) - \gamma \langle Ax, x \rangle^p]$$
$$= \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) - \gamma [\langle A^p x, x \rangle - \langle Ax, x \rangle^p]$$

which is equivalent with the first inequality in (2.111).

By defining the function $g_{\Gamma,p} : (m, M) \to \mathbb{R}$ given by $g_{\Gamma,p}(t) = \Gamma t^p - f(t)$ and applying the same argument we deduce the second part of (2.111).

The rest goes likewise and the details are omitted. ∎

Remark 83 *We observe that if f is a twice differentiable function on (m, M) and $\varphi := \inf_{t \in (m,M)} f''(t), \Phi := \sup_{t \in (m,M)} f''(t)$, then by (2.111) we get the inequality*

$$\frac{1}{2} \varphi \left[\langle A^2 x, x \rangle - \langle Ax, x \rangle^2 \right] \le \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \tag{2.114}$$

$$\le \frac{1}{2} \Phi \left[\langle A^2 x, x \rangle - \langle Ax, x \rangle^2 \right]$$

for each $x \in H$ with $\|x\| = 1$.

We observe that the inequality (2.114) holds for selfadjoint operators that are not necessarily positive.

The following version for sequences of operators can be stated:

Corollary 84 (Dragomir, 2008, [11]) *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, \ldots, n\}$. If f is a twice differentiable function on (m, M) and for $p \in (-\infty, 0) \cup (1, \infty)$ we have the condition (2.110), then*

$$\gamma \left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^p \right] \tag{2.115}$$

$$\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)$$

$$\leq \Gamma \left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^p \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If we have the condition (2.112) for $p \in (0, 1)$, then

$$\delta \left[\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^p - \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \right] \tag{2.116}$$

$$\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)$$

$$\leq \Delta \left[\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^p - \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. Follows from Theorem 82. ∎

Corollary 85 (Dragomir, 2008, [11]) *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, \ldots, n\}$ and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If f is a twice differentiable function on*

(m, M) *and for* $p \in (-\infty, 0) \cup (1, \infty)$ *we have the condition (2.110), then*

$$\gamma \left[\left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^p \right] \tag{2.117}$$

$$\leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right)$$

$$\leq \Gamma \left[\left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^p \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If we have the condition (2.112) for $p \in (0, 1)$, *then*

$$\delta \left[\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^p - \left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \right] \tag{2.118}$$

$$\leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right)$$

$$\leq \Delta \left[\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^p - \left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. Follows from Corollary 84 on choosing $x_j = \sqrt{p_j} \cdot x, \ j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ∎

Remark 86 *We observe that if* f *is a twice differentiable function on* (m, M) *with* $-\infty < m < M < \infty$, $Sp(A_j) \subset [m, M], \ j \in \{1, \ldots, n\}$ *and* $\varphi := \inf_{t \in (m,M)} f''(t), \ \Phi := \sup_{t \in (m,M)} f''(t)$, *then*

$$\varphi \left[\sum_{j=1}^{n} \langle A_j^2 x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^2 \right] \tag{2.119}$$

$$\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)$$

$$\leq \Phi \left[\sum_{j=1}^{n} \langle A_j^2 x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^2 \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Also, if $p_j \geq 0$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, then

$$\varphi \left[\left\langle \sum_{j=1}^{n} p_j A_j^2 x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^2 \right] \tag{2.120}$$

$$\leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right)$$

$$\leq \Phi \left[\left\langle \sum_{j=1}^{n} p_j A_j^2 x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^2 \right]$$

The next result provides some inequalities for the function f which replace the cases $p = 0$ and $p = 1$ that were not allowed in Theorem 82:

Theorem 87 (Dragomir, 2008, [11]) *Let A be a positive definite operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If f is a twice differentiable function on (m, M) and we have for some $\gamma < \Gamma$ that*

$$\gamma \leq t^2 \cdot f''(t) \leq \Gamma \quad \text{for any} \quad t \in (m, M), \tag{2.121}$$

then

$$\gamma \left(\ln (\langle Ax, x \rangle) - \langle \ln Ax, x \rangle \right) \leq \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \tag{2.122}$$
$$\leq \Gamma \left(\ln (\langle Ax, x \rangle) - \langle \ln Ax, x \rangle \right)$$

for each $x \in H$ with $\|x\| = 1$.

If
$$\delta \leq t \cdot f''(t) \leq \Delta \quad \text{for any} \quad t \in (m, M) \tag{2.123}$$
for some $\delta < \Delta$, then

$$\delta \left(\langle A \ln Ax, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \right) \tag{2.124}$$
$$\leq \langle f(A) x, x \rangle - f(\langle Ax, x \rangle)$$
$$\leq \Delta \left(\langle A \ln Ax, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \right)$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Consider the function $g_{\gamma,0} : (m, M) \to \mathbb{R}$ given by $g_{\gamma,0}(t) = f(t) + \gamma \ln t$. The function $g_{\gamma,0}$ is twice differentiable,

$$g''_{\gamma,p}(t) = f''(t) - \gamma t^{-2}$$

for any $t \in (m, M)$ and by (2.121) we deduce that $g_{\gamma,0}$ is convex on (m, M). Now, applying the Mond & Pečarić inequality for $g_{\gamma,0}$ we have

$$0 \leq \langle (f(A) + \gamma \ln A) x, x \rangle - [f(\langle Ax, x \rangle) + \gamma \ln (\langle Ax, x \rangle)]$$
$$= \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) - \gamma [\ln (\langle Ax, x \rangle) - \langle \ln Ax, x \rangle]$$

which is equivalent with the first inequality in (2.122).

By defining the function $g_{\Gamma,0} : (m, M) \to \mathbb{R}$ given by $g_{\Gamma,0}(t) = -\Gamma \ln t - f(t)$ and applying the same argument we deduce the second part of (2.122). The rest goes likewise for the functions

$$g_{\delta,1}(t) = f(t) - \delta t \ln t \quad \text{and} \quad g_{\Delta,0}(t) = \Delta t \ln t - f(t)$$

and the details are omitted. ∎

Corollary 88 (Dragomir, 2008, [11]) *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, \ldots, n\}$. If f is a twice differentiable function on (m, M) and we have the condition (2.121), then*

$$\gamma \left(\ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right) \quad (2.125)$$

$$\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)$$

$$\leq \Gamma \left(\ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.
If we have the condition (2.123), then

$$\delta \left(\sum_{j=1}^{n} \langle A_j \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right) \quad (2.126)$$

$$\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)$$

$$\leq \Delta \left(\sum_{j=1}^{n} \langle A_j \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

The following particular case also holds:

Corollary 89 (Dragomir, 2008, [11]) *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, \ldots, n\}$ and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If f is a twice differentiable function on*

(m, M) *and we have the condition (2.121), then*

$$\gamma \left(\ln \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) - \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle \right) \qquad (2.127)$$

$$\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)$$

$$\leq \Gamma \left(\ln \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) - \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle \right)$$

for each $x \in H$ with $\|x\| = 1$.
  If we have the condition (2.123), then

$$\delta \left(\left\langle \sum_{j=1}^{n} p_j A_j \ln A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \ln \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \right)$$

$$\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \qquad (2.128)$$

$$\leq \Delta \left(\left\langle \sum_{j=1}^{n} p_j A_j \ln A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \ln \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \right)$$

for each $x \in H$ with $\|x\| = 1$.

2.5.2   Applications

It is clear that the results from the previous section can be applied for various particular functions which are twice differentiable and the second derivatives satisfy the boundedness conditions from the statements of the Theorems 82, 87 and the Remark 83.

  We point out here only some simple examples that are, in our opinion, of large interest.

  1. For a given $\alpha > 0$, consider the function $f(t) = \exp(\alpha t), t \in \mathbb{R}$. We have $f''(t) = \alpha^2 \exp(\alpha t)$ and for a selfadjoint operator A with $Sp(A) \subset [m, M]$ (for some real numbers $m < M$) we also have

$$\varphi := \inf_{t \in (m,M)} f''(t) = \alpha^2 \exp(\alpha m) \text{ and } \Phi := \sup_{t \in (m,M)} f''(t) = \alpha^2 \exp(\alpha M).$$

Utilising the inequality (2.114) we get

$$\frac{1}{2}\alpha^2 \exp\left(\alpha m\right)\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right] \tag{2.129}$$
$$\leq \langle \exp\left(\alpha A\right)x, x\rangle - \exp\left(\langle \alpha Ax, x\rangle\right)$$
$$\leq \frac{1}{2}\alpha^2 \exp\left(\alpha M\right)\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right],$$

for each $x \in H$ with $\|x\| = 1$.

Now, if $\beta > 0$, then we also have

$$\frac{1}{2}\beta^2 \exp(-\beta M)\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right] \tag{2.130}$$
$$\leq \langle \exp(-\beta A)x, x\rangle - \exp\left(-\langle \beta Ax, x\rangle\right)$$
$$\leq \frac{1}{2}\beta^2 \exp(-\beta m)\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right],$$

for each $x \in H$ with $\|x\| = 1$.

2. Now, assume that $0 < m < M$ and the operator A satisfies the condition $m \cdot 1_H \leq A \leq M \cdot 1_H$. If we consider the function $f : (0, \infty) \to (0, \infty)$ defined by $f(t) = t^p$ with $p \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$. Then $f''(t) = p(p-1)t^{p-2}$ and if we consider $\varphi := \inf_{t\in(m,M)} f''(t)$ and $\Phi := \sup_{t\in(m,M)} f''(t)$, then we have

$$\varphi = p(p-1)m^{p-2}, \Phi = p(p-1)M^{p-2} \quad \text{for } p \in [2, \infty),$$
$$\varphi = p(p-1)M^{p-2}, \Phi = p(p-1)m^{p-2} \quad \text{for } p \in (1, 2),$$
$$\varphi = p(p-1)m^{p-2}, \Phi = p(p-1)M^{p-2} \quad \text{for } p \in (0, 1),$$

and

$$\varphi = p(p-1)M^{p-2}, \Phi = p(p-1)m^{p-2} \quad \text{for } p \in (-\infty, 0).$$

Utilising the inequality (2.114) we then get the following refinements an reverses of Hölder-McCarthy's inequalities:

$$\frac{1}{2}p(p-1)m^{p-2}\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right] \tag{2.131}$$
$$\leq \langle A^p x, x\rangle - \langle Ax, x\rangle^p$$
$$\leq \frac{1}{2}p(p-1)M^{p-2}\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right] \quad \text{for } p \in [2, \infty),$$

$$\frac{1}{2}p(p-1)M^{p-2}\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right] \tag{2.132}$$
$$\leq \langle A^p x, x\rangle - \langle Ax, x\rangle^p$$
$$\leq \frac{1}{2}p(p-1)m^{p-2}\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right] \quad \text{for } p \in (1, 2),$$

$$\frac{1}{2}p\left(1-p\right)M^{p-2}\left[\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right] \tag{2.133}$$

$$\leq \langle Ax, x\rangle^p - \langle A^px, x\rangle$$

$$\leq \frac{1}{2}p\left(1-p\right)m^{p-2}\left[\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right] \quad \text{for } p \in (0, 1)$$

and

$$\frac{1}{2}p\left(p-1\right)M^{p-2}\left[\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right] \tag{2.134}$$

$$\leq \langle A^px, x\rangle - \langle Ax, x\rangle^p$$

$$\leq \frac{1}{2}p\left(p-1\right)m^{p-2}\left[\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right] \quad \text{for } p \in (-\infty, 0),$$

for each $x \in H$ with $\|x\| = 1$.

3. Now, if we consider the function $f : (0, \infty) \to \mathbb{R}$, $f(t) = -\ln t$, then $f''(t) = t^{-2}$ which gives that $\varphi = M^{-2}$ and $\Phi = m^{-2}$. Utilising the inequality (2.114) we then deduce the bounds

$$\frac{1}{2}M^{-2}\left[\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right] \tag{2.135}$$

$$\leq \ln\left(\langle Ax, x\rangle\right) - \langle \ln Ax, x\rangle$$

$$\leq \frac{1}{2}m^{-2}\left[\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right]$$

for each $x \in H$ with $\|x\| = 1$.

Moreover, if we consider the function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t\ln t$, then $f''(t) = t^{-1}$ which gives that $\varphi = M^{-1}$ and $\Phi = m^{-1}$. Utilising the inequality (2.114) we then deduce the bounds

$$\frac{1}{2}M^{-1}\left[\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right] \tag{2.136}$$

$$\leq \langle A\ln Ax, x\rangle - \langle Ax, x\rangle \ln\left(\langle Ax, x\rangle\right)$$

$$\leq \frac{1}{2}m^{-1}\left[\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right]$$

for each $x \in H$ with $\|x\| = 1$.

Remark 90 *Utilising Theorem 82 for the particular value of $p = -1$ we can state the inequality*

$$\frac{1}{2}\psi\left(\langle A^{-1}x, x\rangle - \langle Ax, x\rangle^{-1}\right) \leq \langle f(A)x, x\rangle - f\left(\langle Ax, x\rangle\right) \tag{2.137}$$

$$\leq \frac{1}{2}\Psi\left(\langle A^{-1}x, x\rangle - \langle Ax, x\rangle^{-1}\right)$$

for each $x \in H$ with $\|x\| = 1$, provided that f is twice differentiable on $(m, M) \subset (0, \infty)$ and

$$\psi = \inf_{t \in (m,M)} t^3 f''(t) \quad \text{while} \quad \Psi = \sup_{t \in (m,M)} t^3 f''(t)$$

are assumed to be finite.

We observe that, by utilising the inequality (2.137) instead of the inequality (2.114) we may obtain similar results in terms of the quantity $\langle A^{-1}x, x\rangle - \langle Ax, x\rangle^{-1}$, $x \in H$ with $\|x\| = 1$. However the details are left to the interested reader.

2.6   Some Jensen's Type Inequalities for Log-convex Functions

2.6.1   Preliminary Results

The following result that provides an operator version for the Jensen inequality for convex functions is due to Mond and Pečarić [32] (see also [20, p. 5]):

Theorem 91 (Mond-Pečarić, 1993, [32]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \le \langle f(A)x, x\rangle \qquad \text{(MP)}$$

for each $x \in H$ with $\|x\| = 1$.

Taking into account the above result and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of *log-convex functions*, namely functions $f : I \to (0, \infty)$ for which $\ln f$ is convex.

We observe that such functions satisfy the elementary inequality

$$f((1 - t)a + tb) \le [f(a)]^{1-t}[f(b)]^{t} \qquad (2.138)$$

for any $a, b \in I$ and $t \in [0, 1]$. Also, due to the fact that the weighted geometric mean is less than the weighted arithmetic mean, it follows that any log-convex function is a convex functions. However, obviously, there are functions that are convex but not log-convex.

As an immediate consequence of the Mond-Pečarić inequality above we can provide the following result:

Theorem 92 (Dragomir, 2010, [15]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$g(\langle Ax, x\rangle) \le \exp\langle \ln g(A)x, x\rangle \le \langle g(A)x, x\rangle \qquad (2.139)$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Consider the function $f := \ln g$, which is convex on $[m, M]$. Writing (MP) for f we get $\ln[g(\langle Ax, x \rangle)] \leq \langle \ln g(A)x, x \rangle$, for each $x \in H$ with $\|x\| = 1$, which, by taking the exponential, produces the first inequality in (2.139).

If we also use (MP) for the exponential function, we get

$$\exp \langle \ln g(A)x, x \rangle \leq \langle \exp[\ln g(A)]x, x \rangle = \langle g(A)x, x \rangle$$

for each $x \in H$ with $\|x\| = 1$ and the proof is complete. ∎

The case of sequences of operators may be of interest and is embodied in the following corollary:

Corollary 93 (Dragomir, 2010, [15]) *Assume that g is as in the Theorem 92. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M], j \in \{1, ..., n\}$ and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then*

$$g\left(\sum_{j=1}^n \langle A_j x_j, x_j \rangle\right) \leq \exp \left\langle \sum_{j=1}^n \ln g(A_j) x_j, x_j \right\rangle \tag{2.140}$$

$$\leq \left\langle \sum_{j=1}^n g(A_j) x_j, x_j \right\rangle.$$

Proof. Follows from Theorem 92 and we omit the details. ∎

In particular we have:

Corollary 94 (Dragomir, 2010, [15]) *Assume that g is as in the Theorem 92. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}, j \in \{1, ..., n\}$ and $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then*

$$g\left(\left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle\right) \leq \left\langle \prod_{j=1}^n [g(A_j)]^{p_j} x, x \right\rangle \tag{2.141}$$

$$\leq \left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Follows from Corollary 93 by choosing $x_j = \sqrt{p_j} \cdot x, j \in \{1, ..., n\}$ where $x \in H$ with $\|x\| = 1$. ∎

It is also important to observe that, as a special case of Theorem 91 we have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality:

Theorem 95 (Hölder-McCarthy, 1967, [26]) *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
(ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
(iii) If A is invertible, then $\langle A^{-r} x, x \rangle \geq \langle Ax, x \rangle^{-r}$ for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

Since the function $g(t) = t^{-r}$ for $r > 0$ is log-convex, we can improve the Hölder-McCarthy inequality as follows:

Proposition 96 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$\langle Ax, x \rangle^{-r} \leq \exp \langle \ln \left(A^{-r} \right) x, x \rangle \leq \langle A^{-r} x, x \rangle \qquad (2.142)$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [20, p. 57]:

Theorem 97 *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$\langle f(A) x, x \rangle \leq \frac{M - \langle Ax, x \rangle}{M - m} \cdot f(m) + \frac{\langle Ax, x \rangle - m}{M - m} \cdot f(M) \qquad (2.143)$$

for each $x \in H$ with $\|x\| = 1$.

This result can be improved for log-convex functions as follows:

Theorem 98 (Dragomir, 2010, [15]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$\langle g(A) x, x \rangle \leq \left\langle \left[[g(m)]^{\frac{M 1_H - A}{M - m}} [g(M)]^{\frac{A - m 1_H}{M - m}} \right] x, x \right\rangle \qquad (2.144)$$

$$\leq \frac{M - \langle Ax, x \rangle}{M - m} \cdot g(m) + \frac{\langle Ax, x \rangle - m}{M - m} \cdot g(M)$$

and

$$g(\langle Ax, x \rangle) \leq [g(m)]^{\frac{M - \langle Ax, x \rangle}{M - m}} [g(M)]^{\frac{\langle Ax, x \rangle - m}{M - m}} \qquad (2.145)$$

$$\leq \left\langle \left[[g(m)]^{\frac{M 1_H - A}{M - m}} [g(M)]^{\frac{A - m 1_H}{M - m}} \right] x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Observe that, by the log-convexity of g, we have

$$g(t) = g \left(\frac{M - t}{M - m} \cdot m + \frac{t - m}{M - m} \cdot M \right) \qquad (2.146)$$

$$\leq [g(m)]^{\frac{M - t}{M - m}} [g(M)]^{\frac{t - m}{M - m}}$$

for any $t \in [m, M]$.

Applying the property (P) for the operator A, we have that

$$\langle g(A) x, x \rangle \leq \langle \Psi(A) x, x \rangle$$

for each $x \in H$ with $\|x\| = 1$, where $\Psi(t) := [g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}}$, $t \in [m, M]$. This proves the first inequality in (2.144).

Now, observe that, by the weighted arithmetic mean-geometric mean inequality we have

$$[g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}} \leq \frac{M-t}{M-m} \cdot g(m) + \frac{t-m}{M-m} \cdot g(M)$$

for any $t \in [m, M]$.

Applying the property (P) for the operator A we deduce the second inequality in (2.144).

Further on, if we use the inequality (2.146) for $t = \langle Ax, x \rangle \in [m, M]$ then we deduce the first part of (2.145).

Now, observe that the function Ψ introduced above can be rearranged to read as

$$\Psi(t) = g(m) \left[\frac{g(M)}{g(m)} \right]^{\frac{t-m}{M-m}}, t \in [m, M]$$

showing that Ψ is a convex function on $[m, M]$.

Applying Mond-Pečarić's inequality for Ψ we deduce the second part of (2.145) and the proof is complete. ∎

The case of sequences of operators is as follows:

Corollary 99 (Dragomir, 2010, [15]) *Assume that g is as in the Theorem 92. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then*

$$\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \tag{2.147}$$

$$\leq \left\langle \sum_{j=1}^{n} \left[[g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} \right] x_j, x_j \right\rangle$$

$$\leq \frac{M - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle}{M - m} \cdot g(m) + \frac{\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - m}{M - m} \cdot g(M)$$

and

$$g \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \tag{2.148}$$

$$\leq [g(m)]^{\frac{M - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle}{M - m}} [g(M)]^{\frac{\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - m}{M - m}}$$

$$\leq \left\langle \sum_{j=1}^{n} \left[[g(m)]^{\frac{M 1_H - A_j}{M - m}} [g(M)]^{\frac{A_j - m 1_H}{M - m}} \right] x_j, x_j \right\rangle.$$

In particular we have:

Corollary 100 (Dragomir, 2010, [15]) *Assume that g is as in the Theorem 92. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \tag{2.149}$$

$$\leq \left\langle \sum_{j=1}^{n} p_j [g(m)]^{\frac{M 1_H - A_j}{M - m}} [g(M)]^{\frac{A_j - m 1_H}{M - m}} x, x \right\rangle$$

$$\leq \frac{M - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle}{M - m} \cdot g(m) + \frac{\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - m}{M - m} \cdot g(M)$$

and

$$g \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \tag{2.150}$$

$$\leq [g(m)]^{\frac{M - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle}{M - m}} [g(M)]^{\frac{\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - m}{M - m}}$$

$$\leq \left\langle \sum_{j=1}^{n} p_j [g(m)]^{\frac{M 1_H - A_j}{M - m}} [g(M)]^{\frac{A_j - m 1_H}{M - m}} x, x \right\rangle.$$

The above result from Theorem 98 can be utilized to produce the following reverse inequality for negative powers of operators:

Proposition 101 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m, M] \, (0 < m < M)$, then*

$$\langle A^{-r} x, x \rangle \leq \left\langle \left[m^{\frac{M 1_H - A}{M - m}} M^{\frac{A - m 1_H}{M - m}} \right]^{-r} x, x \right\rangle \tag{2.151}$$

$$\leq \frac{M - \langle A x, x \rangle}{M - m} \cdot m^{-r} + \frac{\langle A x, x \rangle - m}{M - m} \cdot M^{-r}$$

and

$$\langle Ax, x\rangle^{-r} \le \left[g\left(m\right)^{\frac{M - \langle Ax, x\rangle}{M - m}} g\left(M\right)^{\frac{\langle Ax, x\rangle - m}{M - m}} \right]^{-r} \qquad (2.152)$$

$$\le \left\langle \left[m^{\frac{M 1_H - A}{M - m}} M^{\frac{A - m 1_H}{M - m}} \right]^{-r} x, x \right\rangle$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

2.6.2   Jensen's Inequality for Differentiable Log-convex Functions

The following result provides a reverse for the Jensen type inequality (MP):

Theorem 102 (Dragomir, 2008, [9]) *Let J be an interval and $f : J \to \mathbb{R}$ be a convex and differentiable function on $\mathring{J}$ (the interior of J) whose derivative f' is continuous on $\mathring{J}$. If A is a selfadjoint operators on the Hilbert space H with $Sp\left(A\right) \subseteq [m, M] \subset \mathring{J}$, then*

$$(0 \le) \langle f\left(A\right) x, x\rangle - f\left(\langle Ax, x\rangle\right) \qquad (2.153)$$

$$\le \langle f'\left(A\right) Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'\left(A\right) x, x\rangle$$

for any $x \in H$ with $\|x\| = 1$.

The following result may be stated:

Proposition 103 (Dragomir, 2010, [15]) *Let J be an interval and $g : J \to \mathbb{R}$ be a differentiable log-convex function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operator on the Hilbert space H with $Sp\left(A\right) \subseteq [m, M] \subset \mathring{J}$, then*

$$(1 \le) \frac{\exp \langle \ln g\left(A\right) x, x\rangle}{g\left(\langle Ax, x\rangle\right)} \qquad (2.154)$$

$$\le \exp \left[\left\langle g'\left(A\right) \left[g\left(A\right)\right]^{-1} Ax, x \right\rangle - \langle Ax, x\rangle \cdot \left\langle g'\left(A\right) \left[g\left(A\right)\right]^{-1} x, x \right\rangle \right]$$

for each $x \in H$ with $\|x\| = 1$.

Proof. It follows by the inequality (2.153) written for the convex function $f = \ln g$ that

$$\langle \ln g\left(A\right) x, x\rangle \le \ln g\left(\langle Ax, x\rangle\right)$$

$$+ \left\langle g'\left(A\right) \left[g\left(A\right)\right]^{-1} Ax, x \right\rangle - \langle Ax, x\rangle \cdot \left\langle g'\left(A\right) \left[g\left(A\right)\right]^{-1} x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$.

Now, taking the exponential and dividing by $g\left(\langle Ax, x\rangle\right) > 0$ for each $x \in H$ with $\|x\| = 1$, we deduce the desired result (2.154). ∎

Corollary 104 (Dragomir, 2010, [15]) *Assume that g is as in the Proposition 103 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then

$$(1 \leq) \frac{\exp \left\langle \sum_{j=1}^n \ln g(A_j) x_j, x_j \right\rangle}{g \left(\sum_{j=1}^n \langle A_j x, x_j \rangle \right)} \qquad (2.155)$$

$$\leq \exp \left[\left\langle \sum_{j=1}^n g'(A_j) [g(A_j)]^{-1} A_j x_j, x_j \right\rangle \right.$$

$$\left. - \sum_{j=1}^n \langle A_j x_j, x_j \rangle \cdot \sum_{j=1}^n \left\langle g'(A_j) [g(A_j)]^{-1} x_j, x_j \right\rangle \right].$$

If $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then

$$(1 \leq) \frac{\left\langle \prod_{j=1}^n [g(A_j)]^{p_j} x, x \right\rangle}{g \left(\left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \right)} \qquad (2.156)$$

$$\leq \exp \left[\left\langle \sum_{j=1}^n p_j g'(A_j) [g(A_j)]^{-1} A_j x, x \right\rangle \right.$$

$$\left. - \sum_{j=1}^n p_j \langle A_j x, x \rangle \cdot \sum_{j=1}^n p_j \left\langle g'(A_j) [g(A_j)]^{-1} x, x \right\rangle \right]$$

for each $x \in H$ with $\|x\| = 1$.

Remark 105 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$(1 \leq) \langle Ax, x \rangle^r \exp \left\langle \ln (A^{-r}) x, x \right\rangle \qquad (2.157)$$
$$\leq \exp \left[r \left(\langle Ax, x \rangle \cdot \langle A^{-1} x, x \rangle - 1 \right) \right]$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following result that provides both a refinement and a reverse of the multiplicative version of Jensen's inequality can be stated as well:

Theorem 106 (Dragomir, 2010, [15]) *Let J be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operators on the Hilbert space H with*

$Sp\,(A) \subseteq [m, M] \subset \mathring{J},$ *then*

$$1 \leq \left\langle \exp\left[\frac{g'\left(\langle Ax, x\rangle\right)}{g\left(\langle Ax, x\rangle\right)}\left(A - \langle Ax, x\rangle\,1_H\right)\right]x, x\right\rangle \tag{2.158}$$

$$\leq \frac{\langle g\,(A)\,x, x\rangle}{g\left(\langle Ax, x\rangle\right)} \leq \left\langle \exp\left[g'\,(A)\,[g\,(A)]^{-1}\left(A - \langle Ax, x\rangle\,1_H\right)\right]x, x\right\rangle$$

for each $x \in H$ *with* $\|x\| = 1,$ *where* 1_H *denotes the identity operator on* $H.$

Proof. It is well known that if $h : J \to \mathbb{R}$ is a convex differentiable function on $\mathring{J},$ then the following *gradient inequality* holds

$$h\,(t) - h\,(s) \geq h'\,(s)\,(t - s)$$

for any $t, s \in \mathring{J}.$

Now, if we write this inequality for the convex function $h = \ln g,$ then we get

$$\ln g\,(t) - \ln g\,(s) \geq \frac{g'\,(s)}{g\,(s)}\,(t - s) \tag{2.159}$$

which is equivalent with

$$g\,(t) \geq g\,(s)\exp\left[\frac{g'\,(s)}{g\,(s)}\,(t - s)\right] \tag{2.160}$$

for any $t, s \in \mathring{J}.$

Further, if we take $s := \langle Ax, x\rangle \in [m, M] \subset \mathring{J},$ for a fixed $x \in H$ with $\|x\| = 1,$ in the inequality (2.160), then we get

$$g\,(t) \geq g\left(\langle Ax, x\rangle\right)\exp\left[\frac{g'\left(\langle Ax, x\rangle\right)}{g\left(\langle Ax, x\rangle\right)}\,(t - \langle Ax, x\rangle)\right]$$

for any $t \in \mathring{J}.$

Utilising the property (P) for the operator A and the Mond-Pečarić inequality for the exponential function, we can state the following inequality that is of interest in itself as well:

$$\langle g\,(A)\,y, y\rangle \tag{2.161}$$

$$\geq g\left(\langle Ax, x\rangle\right)\left\langle \exp\left[\frac{g'\left(\langle Ax, x\rangle\right)}{g\left(\langle Ax, x\rangle\right)}\left(A - \langle Ax, x\rangle\,1_H\right)\right]y, y\right\rangle$$

$$\geq g\left(\langle Ax, x\rangle\right)\exp\left[\frac{g'\left(\langle Ax, x\rangle\right)}{g\left(\langle Ax, x\rangle\right)}\left(\langle Ay, y\rangle - \langle Ax, x\rangle\right)\right]$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1.$

Further, if we put $y = x$ in (2.161), then we deduce the first and the second inequality in (2.158).

Now, if we replace s with t in (2.160) we can also write the inequality

$$g(t) \exp\left[\frac{g'(t)}{g(t)}(s-t)\right] \leq g(s)$$

which is equivalent with

$$g(t) \leq g(s) \exp\left[\frac{g'(t)}{g(t)}(t-s)\right] \qquad (2.162)$$

for any $t, s \in \mathring{J}$.

Further, if we take $s := \langle Ax, x \rangle \in [m, M] \subset \mathring{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in the inequality (2.162), then we get

$$g(t) \leq g(\langle Ax, x \rangle) \exp\left[\frac{g'(t)}{g(t)}(t - \langle Ax, x \rangle)\right]$$

for any $t \in \mathring{J}$.

Utilising the property (P) for the operator A, then we can state the following inequality that is of interest in itself as well:

$$\langle g(A) y, y \rangle \qquad (2.163)$$
$$\leq g(\langle Ax, x \rangle) \left\langle \exp\left[g'(A) [g(A)]^{-1} (A - \langle Ax, x \rangle 1_H)\right] y, y \right\rangle$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, if we put $y = x$ in (2.163), then we deduce the last inequality in (2.158). ∎

The case of operator sequences is embodied in the following corollary:

Corollary 107 (Dragomir, 2010, [15]) *Assume that g is as in the Proposition 103 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

$$1 \qquad (2.164)$$

$$\leq \left\langle \sum_{j=1}^{n} \exp\left[\frac{g'\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)}{g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)} \left(A_j - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle 1_H\right)\right] x_j, x_j \right\rangle$$

$$\leq \frac{\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle}{g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)}$$

$$\leq \left\langle \sum_{j=1}^{n} \exp\left[g'(A_j) [g(A_j)]^{-1} \left(A_j - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle 1_H\right)\right] x_j, x_j \right\rangle.$$

If $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then for each $x \in H$ with $\|x\| = 1$

$$1 \leq \left\langle \sum_{j=1}^{n} p_j \exp \left[\frac{g'\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right)}{g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right)} \right. \right. \tag{2.165}$$

$$\left. \left. \times \left(A_j - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle 1_H \right) \right] x, x \right\rangle$$

$$\leq \frac{\left\langle \sum_{j=1}^{n} p_j g\left(A_j\right) x, x \right\rangle}{g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right)}$$

$$\leq \left\langle \sum_{j=1}^{n} p_j \exp \left[g'\left(A_j\right) \left[g\left(A_j\right)\right]^{-1} \left(A_j - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle 1_H \right) \right] x, x \right\rangle.$$

Remark 108 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$1 \leq \left\langle \exp \left[r \left(1_H - \left\langle Ax, x \right\rangle^{-1} A \right) \right] x, x \right\rangle \tag{2.166}$$

$$\leq \left\langle A^{-r} x, x \right\rangle \left\langle Ax, x \right\rangle^{r} \leq \left\langle \exp \left[r \left(1_H - \left\langle Ax, x \right\rangle A^{-1} \right) \right] x, x \right\rangle$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following reverse inequality may be proven as well:

Theorem 109 (Dragomir, 2010, [15]) *Let J be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$(1 \leq) \frac{\left\langle \left[g(M)\right]^{\frac{A - m1_H}{M - m}} \left[g(m)\right]^{\frac{M1_H - A}{M - m}} x, x \right\rangle}{\left\langle g(A) x, x \right\rangle} \tag{2.167}$$

$$\leq \frac{\left\langle g(A) \exp \left[\frac{(M1_H - A)(A - m1_H)}{M - m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right] x, x \right\rangle}{\left\langle g(A) x, x \right\rangle}$$

$$\leq \exp \left[\frac{1}{4} (M - m) \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Utilising the inequality (2.159) we have successively

$$\frac{g((1 - \lambda)t + \lambda s)}{g(s)} \geq \exp \left[(1 - \lambda) \frac{g'(s)}{g(s)} (t - s) \right] \tag{2.168}$$

and

$$\frac{g\left((1-\lambda)t+\lambda s\right)}{g\left(t\right)} \geq \exp\left[-\lambda\frac{g'\left(t\right)}{g\left(t\right)}\left(t-s\right)\right] \qquad (2.169)$$

for any $t, s \in \overset{\circ}{J}$ and any $\lambda \in [0,1]$.

Now, if we take the power λ in the inequality (2.168) and the power $1-\lambda$ in (2.169) and multiply the obtained inequalities, we deduce

$$\frac{\left[g\left(t\right)\right]^{1-\lambda}\left[g\left(s\right)\right]^{\lambda}}{g\left((1-\lambda)t+\lambda s\right)} \qquad (2.170)$$

$$\leq \exp\left[(1-\lambda)\lambda\left(\frac{g'\left(t\right)}{g\left(t\right)}-\frac{g'\left(s\right)}{g\left(s\right)}\right)\left(t-s\right)\right]$$

for any $t, s \in \overset{\circ}{J}$ and any $\lambda \in [0,1]$.

Further on, if we choose in (2.170) $t = M, s = m$ and $\lambda = \frac{M-u}{M-m}$, then, from (2.170) we get the inequality

$$\frac{\left[g\left(M\right)\right]^{\frac{u-m}{M-m}}\left[g\left(m\right)\right]^{\frac{M-u}{M-m}}}{g\left(u\right)} \qquad (2.171)$$

$$\leq \exp\left[\frac{\left(M-u\right)\left(u-m\right)}{M-m}\left(\frac{g'\left(M\right)}{g\left(M\right)}-\frac{g'\left(m\right)}{g\left(m\right)}\right)\right]$$

which, together with the inequality

$$\frac{\left(M-u\right)\left(u-m\right)}{M-m} \leq \frac{1}{4}\left(M-m\right)$$

produce

$$\left[g\left(M\right)\right]^{\frac{u-m}{M-m}}\left[g\left(m\right)\right]^{\frac{M-u}{M-m}} \qquad (2.172)$$

$$\leq g\left(u\right)\exp\left[\frac{\left(M-u\right)\left(u-m\right)}{M-m}\left(\frac{g'\left(M\right)}{g\left(M\right)}-\frac{g'\left(m\right)}{g\left(m\right)}\right)\right]$$

$$\leq g\left(u\right)\exp\left[\frac{1}{4}\left(M-m\right)\left(\frac{g'\left(M\right)}{g\left(M\right)}-\frac{g'\left(m\right)}{g\left(m\right)}\right)\right]$$

for any $u \in [m, M]$.

If we apply the property (P) to the inequality (2.172) and for the operator A we deduce the desired result. ∎

Corollary 110 (Dragomir, 2010, [15]) *Assume that g is as in the Theorem 109 and A_j are selfadjoint operators with $Sp\left(A_j\right) \subseteq [m, M] \subset \overset{\circ}{J}$, $j \in \{1, ..., n\}$.*

If $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

$$(1 \leq) \frac{\sum_{j=1}^{n} \left\langle [g(M)]^{\frac{A_j - m1_H}{M-m}} [g(m)]^{\frac{M1_H - A_j}{M-m}} x_j, x_j \right\rangle}{\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} \tag{2.173}$$

$$\leq \frac{\sum_{j=1}^{n} \left\langle g(A_j) \exp\left[\frac{(M1_H - A_j)(A_j - m1_H)}{M-m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right] x_j, x_j \right\rangle}{\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle}$$

$$\leq \exp\left[\frac{1}{4} (M-m) \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right].$$

If $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then for each $x \in H$ with $\|x\| = 1$

$$(1 \leq) \frac{\left\langle \sum_{j=1}^{n} p_j [g(M)]^{\frac{A_j - m1_H}{M-m}} [g(m)]^{\frac{M1_H - A_j}{M-m}} x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle} \tag{2.174}$$

$$\leq \frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) \exp\left[\frac{(M1_H - A_j)(A_j - m1_H)}{M-m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right] x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle}$$

$$\leq \exp\left[\frac{1}{4} (M-m) \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right].$$

Remark 111 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m, M]$ $(0 < m < M)$, then*

$$(1 \leq) \frac{\left\langle [g(M)]^{\frac{r(m1_H - A)}{M-m}} [g(m)]^{\frac{r(A - M1_H)}{M-m}} x, x \right\rangle}{\langle A^{-r} x, x \rangle} \tag{2.175}$$

$$\leq \frac{\left\langle A^{-r} \exp\left[\frac{r(M1_H - A)(A - m1_H)}{Mm} \right] x, x \right\rangle}{\langle A^{-r} x, x \rangle} \leq \exp\left[\frac{1}{4} r \frac{(M-m)^2}{mM} \right]$$

2.6.3 Applications for Ky Fan's Inequality

Consider the function $g : (0, 1) \to \mathbb{R}$, $g(t) = \left(\frac{1-t}{t} \right)^r$, $r > 0$. Observe that for the new function $f : (0, 1) \to \mathbb{R}$, $f(t) = \ln g(t)$ we have

$$f'(t) = \frac{-r}{t(1-t)} \text{ and } f''(t) = \frac{2r\left(\frac{1}{2} - t\right)}{t^2(1-t)^2} \text{ for } t \in (0, 1)$$

showing that the function g is log-convex on the interval $\left(0, \frac{1}{2}\right)$.

If $p_i > 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $t_i \in \left(0, \frac{1}{2}\right)$ for $i \in \{1, ..., n\}$, then by applying the Jensen inequality for the convex function f (with $r = 1$) on the interval $\left(0, \frac{1}{2}\right)$ we get

$$\frac{\sum_{i=1}^{n} p_i t_i}{1 - \sum_{i=1}^{n} p_i t_i} \geq \prod_{i=1}^{n} \left(\frac{t_i}{1 - t_i}\right)^{p_i}, \tag{2.176}$$

which is the weighted version of the celebrated *Ky Fan's inequality*, see [3, p. 3].

This inequality is equivalent with

$$\prod_{i=1}^{n} \left(\frac{1 - t_i}{t_i}\right)^{p_i} \geq \frac{1 - \sum_{i=1}^{n} p_i t_i}{\sum_{i=1}^{n} p_i t_i},$$

where $p_i > 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $t_i \in \left(0, \frac{1}{2}\right)$ for $i \in \{1, ..., n\}$.

By the weighted arithmetic mean - geometric mean inequality we also have that

$$\sum_{i=1}^{n} p_i (1 - t_i) t_i^{-1} \geq \prod_{i=1}^{n} \left(\frac{1 - t_i}{t_i}\right)^{p_i}$$

giving the double inequality

$$\sum_{i=1}^{n} p_i (1 - t_i) t_i^{-1} \geq \prod_{i=1}^{n} \left((1 - t_i) t_i^{-1}\right)^{p_i} \tag{2.177}$$

$$\geq \sum_{i=1}^{n} p_i (1 - t_i) \left(\sum_{i=1}^{n} p_i t_i\right)^{-1}.$$

The following operator inequalities generalizing (2.177) may be stated:

Proposition 112 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subset \left(0, \frac{1}{2}\right)$, then*

$$\left\langle \left(A^{-1} (1_H - A)\right)^r x, x \right\rangle \geq \exp \left\langle \ln \left(A^{-1} (1_H - A)\right)^r x, x \right\rangle \tag{2.178}$$

$$\geq \left(\left\langle (1_H - A) x, x \right\rangle \left\langle Ax, x \right\rangle^{-1}\right)^r$$

for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

In particular,

$$\left\langle A^{-1} (1_H - A) x, x \right\rangle \geq \exp \left\langle \ln \left(A^{-1} (1_H - A)\right) x, x \right\rangle \tag{2.179}$$

$$\geq \left\langle (1_H - A) x, x \right\rangle \left\langle Ax, x \right\rangle^{-1}$$

for each $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 92 applied for the log-convex function $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0, t \in \left(0, \frac{1}{2}\right)$.

Proposition 113 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m, M] \subset \left(0, \frac{1}{2}\right)$, then*

$$\left\langle \left((1_H - A) A^{-1}\right)^r x, x \right\rangle \tag{2.180}$$

$$\leq \left\langle \left[\left(\frac{1-m}{m}\right)^{\frac{r(M1_H - A)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(A-m1_H)}{M-m}}\right] x, x \right\rangle$$

$$\leq \frac{M - \langle Ax, x \rangle}{M - m} \cdot \left(\frac{1-m}{m}\right)^r + \frac{\langle Ax, x \rangle - m}{M - m} \cdot \left(\frac{1-M}{M}\right)^r$$

and

$$\left(\frac{1 - \langle Ax, x \rangle}{\langle Ax, x \rangle}\right)^r \tag{2.181}$$

$$\leq \left(\frac{1-m}{m}\right)^{\frac{r(M - \langle Ax, x \rangle)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(\langle Ax, x \rangle - m)}{M-m}}$$

$$\leq \left\langle \left[\left(\frac{1-m}{m}\right)^{\frac{r(M1_H - A)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(A-m1_H)}{M-m}}\right] x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

The proof follows by Theorem 98 applied for the log-convex function $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0, t \in \left(0, \frac{1}{2}\right)$.
Finally we have:

Proposition 114 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subset \left(0, \frac{1}{2}\right)$, then*

$$(1 \leq) \frac{\exp \left\langle \ln \left((1_H - A) A^{-1}\right)^r x, x \right\rangle}{\left((1 - \langle Ax, x \rangle) \langle Ax, x \rangle^{-1}\right)^r} \tag{2.182}$$

$$\leq \exp \left[r \left(\langle Ax, x \rangle \cdot \left\langle A^{-1} (1_H - A)^{-1} x, x \right\rangle - \left\langle (1_H - A)^{-1} x, x \right\rangle\right)\right]$$

and

$$1 \leq \left\langle \exp \left[r (1 - \langle Ax, x \rangle)^{-1} \left(1_H - \langle Ax, x \rangle^{-1} A\right)\right] x, x \right\rangle \tag{2.183}$$

$$\leq \frac{\left\langle \left((1_H - A) A^{-1}\right)^r x, x \right\rangle}{\left((1 - \langle Ax, x \rangle) \langle Ax, x \rangle^{-1}\right)^r}$$

$$\leq \left\langle \exp \left[r (1_H - A)^{-1} \left(\langle Ax, x \rangle A^{-1} - 1_H\right)\right] x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

The proof follows by Proposition 103 and Theorem 106 applied for the log-convex function $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0, t \in \left(0, \frac{1}{2}\right)$. The details are omitted.

2.6.4 More Inequalities for Differentiable Log-convex Functions

The following results providing companion inequalities for the Jensen inequality for differentiable log-convex functions obtained above hold:

Theorem 115 (Dragomir, 2010, [16]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : J \to (0, \infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m, M] \subset \mathring{J}$, then*

$$\exp\left[\frac{\langle g'(A)Ax, x\rangle}{\langle g(A)x, x\rangle} - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle} \cdot \frac{\langle g'(A)x, x\rangle}{\langle g(A)x, x\rangle}\right] \quad (2.184)$$

$$\geq \frac{\exp\left[\frac{\langle g(A)\ln g(A)x, x\rangle}{\langle g(A)x, x\rangle}\right]}{g\left(\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\right)} \geq 1$$

for each $x \in H$ with $\|x\| = 1$.
    If

$$\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle} \in \mathring{J} \text{ for each } x \in H \text{ with } \|x\| = 1, \quad (C)$$

then

$$\exp\left[\frac{g'\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right)}\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle} - \frac{\langle Ag(A)x, x\rangle}{\langle g(A)x, x\rangle}\right)\right] \quad (2.185)$$

$$\geq \frac{g\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right)}{\exp\left(\frac{\langle g(A)\ln g(A)x, x\rangle}{\langle g(A)x, x\rangle}\right)} \geq 1,$$

for each $x \in H$ with $\|x\| = 1$.

Proof. By the gradient inequality for the convex function $\ln g$ we have

$$\frac{g'(t)}{g(t)}(t - s) \geq \ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)}(t - s) \quad (2.186)$$

for any $t, s \in \mathring{J}$, which by multiplication with $g(t) > 0$ is equivalent with

$$g'(t)(t - s) \geq g(t)\ln g(t) - g(t)\ln g(s) \geq \frac{g'(s)}{g(s)}(tg(t) - sg(t)) \quad (2.187)$$

for any $t, s \in \mathring{J}$.

Fix $s \in \mathring{J}$ and apply the property (P) to get that

$$\langle g'(A)Ax, x\rangle - s\langle g'(A)x, x\rangle \tag{2.188}$$
$$\geq \langle g(A)\ln g(A)x, x\rangle - \langle g(A)x, x\rangle \ln g(s)$$
$$\geq \frac{g'(s)}{g(s)}\left(\langle Ag(A)x, x\rangle - s\langle g(A)x, x\rangle\right)$$

for any $x \in H$ with $\|x\| = 1$, which is an inequality of interest in itself as well.

Since

$$\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle} \in [m, M] \text{ for any } x \in H \text{ with } \|x\| = 1$$

then on choosing $s := \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}$ in (2.188) we get

$$\langle g'(A)Ax, x\rangle - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\langle g'(A)x, x\rangle$$
$$\geq \langle g(A)\ln g(A)x, x\rangle - \langle g(A)x, x\rangle \ln g\left(\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\right) \geq 0,$$

which, by division with $\langle g(A)x, x\rangle > 0$, produces

$$\frac{\langle g'(A)Ax, x\rangle}{\langle g(A)x, x\rangle} - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle} \cdot \frac{\langle g'(A)x, x\rangle}{\langle g(A)x, x\rangle} \tag{2.189}$$
$$\geq \frac{\langle g(A)\ln g(A)x, x\rangle}{\langle g(A)x, x\rangle} - \ln g\left(\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\right) \geq 0$$

for any $x \in H$ with $\|x\| = 1$.

Taking the exponential in (2.189) we deduce the desired inequality (2.184).

Now, assuming that the condition (C) holds, then by choosing $s := \frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}$ in (2.188) we get

$$0 \geq \langle g(A)\ln g(A)x, x\rangle - \langle g(A)x, x\rangle \ln g\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right)$$

$$\geq \frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}\left(\langle Ag(A)x, x\rangle - \frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\langle g(A)x, x\rangle\right)$$

which, by dividing with $\langle g(A)x, x\rangle > 0$ and rearranging, is equivalent with

$$\frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle} - \frac{\langle Ag(A)x, x\rangle}{\langle g(A)x, x\rangle}\right) \tag{2.190}$$

$$\geq \ln g\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right) - \frac{\langle g(A)\ln g(A)x, x\rangle}{\langle g(A)x, x\rangle} \geq 0$$

for any $x \in H$ with $\|x\| = 1$.

Finally, on taking the exponential in (2.190) we deduce the desired inequality (2.185). ∎

Remark 116 *We observe that a sufficient condition for (C) to hold is that either $g'(A)$ or $-g'(A)$ is a positive definite operator on H.*

Corollary 117 (Dragomir, 2010, [16]) *Assume that A and g are as in Theorem 115. If the condition (C) holds, then we have the double inequality*

$$\ln g \left(\frac{\langle g'(A) Ax, x \rangle}{\langle g'(A) x, x \rangle} \right) \geq \frac{\langle g(A) \ln g(A) x, x \rangle}{\langle g(A) x, x \rangle} \tag{2.191}$$

$$\geq \ln g \left(\frac{\langle g(A) Ax, x \rangle}{\langle g(A) x, x \rangle} \right),$$

for any $x \in H$ with $\|x\| = 1$.

Remark 118 *Assume that A is a positive definite operator on H. Since for $r > 0$ the function $g(t) = t^{-r}$ is log-convex on $(0, \infty)$ and*

$$\frac{\langle g'(A) Ax, x \rangle}{\langle g'(A) x, x \rangle} = \frac{\langle A^{-r} x, x \rangle}{\langle A^{-r-1} x, x \rangle} > 0$$

for any $x \in H$ with $\|x\| = 1$, then on applying the inequality (2.191) we deduce the following interesting result

$$\ln \left(\frac{\langle A^{-r} x, x \rangle}{\langle A^{-r-1} x, x \rangle} \right) \leq \frac{\langle A^{-r} \ln Ax, x \rangle}{\langle A^{-r} x, x \rangle} \leq \ln \left(\frac{\langle A^{-r+1} x, x \rangle}{\langle A^{-r} x, x \rangle} \right) \tag{2.192}$$

for any $x \in H$ with $\|x\| = 1$.

The details of the proof are left to the interested reader.

The case of sequences of operators is embodied in the following corollary:

Corollary 119 (Dragomir, 2010, [16]) *Let A_j, $j \in \{1, ..., n\}$ be selfadjoint operators on the Hilbert space H and assume that $Sp(A_j) \subseteq [m, M]$ for some scalars m, M with $m < M$ and each $j \in \{1, ..., n\}$. If $g : J \to (0, \infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m, M] \subset \mathring{J}$, then*

$$\exp \left[\frac{\sum_{j=1}^n \langle g'(A_j) A_j x_j, x_j \rangle}{\sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle} \right. \tag{2.193}$$

$$\left. - \frac{\sum_{j=1}^n \langle g(A_j) A_j x_j, x_j \rangle}{\sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle} \cdot \frac{\sum_{j=1}^n \langle g'(A_j) x_j, x_j \rangle}{\sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle} \right]$$

$$\geq \frac{\exp \left[\frac{\sum_{j=1}^n \langle g(A_j) \ln g(A_j) x_j, x_j \rangle}{\sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle} \right]}{g \left(\frac{\sum_{j=1}^n \langle g(A_j) A_j x_j, x_j \rangle}{\sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle} \right)} \geq 1$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

If

$$\frac{\sum_{j=1}^n \langle g'(A_j) A_j x_j, x_j \rangle}{\sum_{j=1}^n \langle g'(A_j) x_j, x_j \rangle} \in \mathring{J} \tag{2.194}$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then

$$\exp \left[\frac{g'\left(\frac{\sum_{j=1}^n \langle g'(A_j) A_j x_j, x_j \rangle}{\sum_{j=1}^n \langle g'(A_j) x_j, x_j \rangle}\right)}{g\left(\frac{\sum_{j=1}^n \langle g'(A_j) A_j x_j, x_j \rangle}{\sum_{j=1}^n \langle g'(A_j) x_j, x_j \rangle}\right)} \tag{2.195}$$

$$\times \left(\frac{\sum_{j=1}^n \langle g'(A_j) A_j x_j, x_j \rangle}{\sum_{j=1}^n \langle g'(A_j) x_j, x_j \rangle} - \frac{\sum_{j=1}^n \langle A_j g(A_j) x_j, x_j \rangle}{\sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle}\right)\right]$$

$$\geq \frac{g\left(\frac{\sum_{j=1}^n \langle g'(A_j) A_j x_j, x_j \rangle}{\sum_{j=1}^n \langle g'(A_j) x_j, x_j \rangle}\right)}{\exp\left(\frac{\sum_{j=1}^n \langle g(A_j) \ln g(A_j) x_j, x_j \rangle}{\sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle}\right)} \geq 1,$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

The following particular case for sequences of operators also holds:

Corollary 120 (Dragomir, 2010, [16]) *With the assumptions of Corollary 119 and if $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then*

$$\exp \left[\frac{\left\langle \sum_{j=1}^n p_j g'(A_j) A_j x, x \right\rangle}{\left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle} \tag{2.196}$$

$$- \frac{\left\langle \sum_{j=1}^n p_j g(A_j) A_j x, x \right\rangle}{\left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle} \cdot \frac{\left\langle \sum_{j=1}^n p_j g'(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle}\right]$$

$$\geq \frac{\exp\left[\frac{\left\langle \sum_{j=1}^n p_j g(A_j) \ln g(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle}\right]}{g\left(\frac{\left\langle \sum_{j=1}^n p_j g(A_j) A_j x, x \right\rangle}{\left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle}\right)} \geq 1$$

for each $x \in H$, with $\|x\| = 1$.

If

$$\frac{\left\langle \sum_{j=1}^n p_j g'(A_j) A_j x, x \right\rangle}{\left\langle \sum_{j=1}^n p_j g'(A_j) x, x \right\rangle} \in \mathring{J} \tag{2.197}$$

for each $x \in H$, with $\|x\| = 1$, then

$$\exp\left[\frac{g'\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j)A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j)x, x\right\rangle}\right)}{g\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j)A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j)x, x\right\rangle}\right)} \right. \tag{2.198}$$

$$\left. \times \left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle} - \frac{\left\langle \sum_{j=1}^{n} p_j A_j g(A_j) x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right)\right]$$

$$\geq \frac{g\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j)A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j)x, x\right\rangle}\right)}{\exp\left(\frac{\left\langle \sum_{j=1}^{n} p_j g(A_j)\ln g(A_j)x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j)x, x\right\rangle}\right)} \geq 1,$$

for each $x \in H$, with $\|x\| = 1$.

Proof. Follows from Corollary 119 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, ..., n\}$ where $x \in H$ with $\|x\| = 1$. ∎

The following result providing different inequalities also holds:

Theorem 121 (Dragomir, 2010, [16]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : J \to (0, \infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m, M] \subset \mathring{J}$, then*

$$\left\langle \exp\left[g'(A)\left(A - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}1_H\right)\right] x, x\right\rangle \tag{2.199}$$

$$\geq \left\langle \left(\frac{g(A)}{g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)}\right)^{g(A)} x, x\right\rangle$$

$$\geq \left\langle \exp\left[\frac{g'\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)}{g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)}\left(Ag(A) - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}g(A)\right)\right] x, x\right\rangle \geq 1$$

for each $x \in H$ with $\|x\| = 1$.

If the condition (C) from Theorem 115 holds, then

$$\left\langle \exp\left[\frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)} \left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle} g(A) - Ag(A)\right)\right] x, x\right\rangle \quad (2.200)$$

$$\geq \left\langle \left(g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)[g(A)]^{-1}\right)^{g(A)} x, x\right\rangle$$

$$\geq \left\langle \exp\left[g'(A)\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle} 1_H - A\right)\right] x, x\right\rangle \geq 1$$

for each $x \in H$ with $\|x\| = 1$.

Proof. By taking the exponential in (2.187) we have the following inequality

$$\exp\left[g'(t)(t-s)\right] \geq \left(\frac{g(t)}{g(s)}\right)^{g(t)} \geq \exp\left[\frac{g'(s)}{g(s)}(tg(t) - sg(t))\right] \quad (2.201)$$

for any $t, s \in \mathring{J}$.

If we fix $s \in \mathring{J}$ and apply the property (P) to the inequality (2.201), we deduce

$$\langle \exp\left[g'(A)(A - s1_H)\right] x, x\rangle \quad (2.202)$$

$$\geq \left\langle \left(\frac{g(A)}{g(s)}\right)^{g(A)} x, x\right\rangle$$

$$\geq \left\langle \exp\left[\frac{g'(s)}{g(s)}(Ag(A) - sg(A))\right] x, x\right\rangle$$

for each $x \in H$ with $\|x\| = 1$, where 1_H is the identity operator on H.

By Mond-Pečarić's inequality applied for the convex function exp we also have

$$\left\langle \exp\left[\frac{g'(s)}{g(s)}(Ag(A) - sg(A))\right] x, x\right\rangle \quad (2.203)$$

$$\geq \exp\left(\frac{g'(s)}{g(s)}(\langle Ag(A)x, x\rangle - s\langle g(A)x, x\rangle)\right)$$

for each $s \in \mathring{J}$ and $x \in H$ with $\|x\| = 1$.

Now, if we choose $s := \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle} \in [m, M]$ in (2.202) and (2.203) we deduce the desired result (2.199).

Observe that, the inequality (2.201) is equivalent with

$$\exp\left[\frac{g'(s)}{g(s)}(sg(t) - tg(t))\right] \geq \left(\frac{g(s)}{g(t)}\right)^{g(t)} \geq \exp\left[g'(t)(s-t)\right] \quad (2.204)$$

for any $t, s \in \mathring{J}$.

If we fix $s \in \mathring{J}$ and apply the property (P) to the inequality (2.204) we deduce

$$\left\langle \exp \left[\frac{g'(s)}{g(s)} \left(sg(A) - Ag(A) \right) \right] x, x \right\rangle \tag{2.205}$$

$$\geq \left\langle \left(g(s) \left[g(A) \right]^{-1} \right)^{g(A)} x, x \right\rangle$$

$$\geq \left\langle \exp \left[g'(A) \left(s1_H - A \right) \right] x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$.

By Mond-Pečarić's inequality we also have

$$\left\langle \exp \left[g'(A) \left(s1_H - A \right) \right] x, x \right\rangle \geq \exp \left[s \left\langle g'(A) x, x \right\rangle - \left\langle g'(A) Ax, x \right\rangle \right] \tag{2.206}$$

for each $s \in \mathring{J}$ and $x \in H$ with $\|x\| = 1$.

Taking into account that the condition (C) is valid, then we can choose in (2.205) and (2.206) $s := \frac{\langle g'(A)Ax,x \rangle}{\langle g'(A)x,x \rangle}$ to get the desired result (2.200). ∎

Remark 122 *If we apply, for instance, the inequality (2.199) for the log-convex function $g(t) = t^{-1}, t > 0$, then, after simple calculations, we get the inequality*

$$\left\langle \exp \left(\frac{A^{-2} - \left\langle A^{-1}x, x \right\rangle A^{-1}}{A^{-2} - \left\langle A^{-1}x, x \right\rangle} \right) x, x \right\rangle \tag{2.207}$$

$$\geq \left\langle \left(\left\langle A^{-1}x, x \right\rangle A^{-1} \right)^{A^{-1}} x, x \right\rangle$$

$$\geq \left\langle \exp \left(\frac{A^{-1} - \left\langle A^{-1}x, x \right\rangle 1_H}{\left\langle A^{-1}x, x \right\rangle^2} \right) x, x \right\rangle$$

$$\geq 1$$

for each $x \in H$ with $\|x\| = 1$.

Other similar results can be obtained from the inequality (2.200), however the details are left to the interested reader.

2.6.5   A Reverse Inequality

The following reverse inequality is also of interest:

Theorem 123 (Dragomir, 2010, [16]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : J \to (0, \infty)$ is a differentiable log-convex function*

with the derivative continuous on $\mathring{J}$ *and* $[m, M] \subset \mathring{J}$, *then*

$$(1 \leq) \frac{[g(m)]^{\frac{M - \langle Ax, x \rangle}{M - m}} [g(M)]^{\frac{\langle Ax, x \rangle - m}{M - m}}}{\exp \langle \ln g(A) x, x \rangle} \tag{2.208}$$

$$\leq \exp \left[\frac{\langle (M1_H - A)(A - m1_H) x, x \rangle}{M - m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

$$\leq \exp \left[\frac{(M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)}{M - m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

$$\leq \exp \left[\frac{1}{4} (M - m) \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. Utilising the inequality (2.186) we have successively

$$\ln g\left((1 - \lambda) t + \lambda s\right) - \ln g(s) \geq (1 - \lambda) \frac{g'(s)}{g(s)} (t - s) \tag{2.209}$$

and

$$\ln g\left((1 - \lambda) t + \lambda s\right) - \ln g(t) \geq -\lambda \frac{g'(t)}{g(t)} (t - s) \tag{2.210}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0, 1]$.

Now, if we multiply (2.209) by λ and (2.210) by $1 - \lambda$ and sum the obtained inequalities, we deduce

$$(1 - \lambda) \ln g(t) + \lambda \ln g(s) - \ln g\left((1 - \lambda) t + \lambda s\right) \tag{2.211}$$

$$\leq (1 - \lambda) \lambda \left[\left(\frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)} \right) (t - s) \right]$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0, 1]$.

Now, if we choose $\lambda := \frac{M - u}{M - m}, s := m$ and $t := M$ in (2.211) then we get the inequality

$$\frac{u - m}{M - m} \ln g(M) + \frac{M - u}{M - m} \ln g(m) - \ln g(u) \tag{2.212}$$

$$\leq \left[\frac{(M - u)(u - m)}{M - m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

for any $u \in [m, M]$.

If we use the property (P) for the operator A we get

$$\frac{\langle Ax, x \rangle - m}{M - m} \ln g(M) + \frac{M - \langle Ax, x \rangle}{M - m} \ln g(m) - \langle \ln g(A) x, x \rangle \tag{2.213}$$

$$\leq \left[\frac{\langle (M1_H - A)(A - m1_H) x, x \rangle}{M - m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

for each $x \in H$ with $\|x\| = 1$.

Taking the exponential in (2.213) we deduce the first inequality in (2.208).

Now, consider the function $h : [m, M] \rightarrow \mathbb{R}$, $h(t) = (M - t)(t - m)$. This function is concave in $[m, M]$ and by Mond-Pečarić's inequality we have

$$\langle (M1_H - A)(A - m1_H) x, x \rangle \leq (M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)$$

for each $x \in H$ with $\|x\| = 1$, which proves the second inequality in (2.208).

For the last inequality, we observe that

$$(M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m) \leq \frac{1}{4}(M - m)^2,$$

and the proof is complete. ∎

Corollary 124 (Dragomir, 2010, [16]) *Assume that g is as in Theorem 123 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

$$(1 \leq) \frac{[g(m)]^{\frac{M - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle}{M - m}} [g(M)]^{\frac{\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - m}{M - m}}}{\exp\left(\sum_{j}^{n} \langle \ln g(A_j) x_j, x_j \rangle\right)} \qquad (2.214)$$

$$\leq \exp\left[\frac{\sum_{j=1}^{n} \langle (M1_H - A_j)(A_j - m1_H) x_j, x_j \rangle}{M - m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$

$$\leq \exp\left[\frac{\left(M - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - m\right)}{M - m}\right.$$

$$\left. \times \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$

$$\leq \exp\left[\frac{1}{4}(M - m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right].$$

If $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then

$$(1 \leq) \frac{[g(m)]^{\frac{M - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle}{M - m}} [g(M)]^{\frac{\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - m}{M - m}}}{\left\langle \prod_{j=1}^{n} [g(A_j)]^{p_j} x, x \right\rangle} \qquad (2.215)$$

$$\leq \exp \left[\frac{\sum_{j=1}^{n} p_j \left\langle (M 1_H - A_j)(A_j - m 1_H) x_j, x_j \right\rangle}{M - m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

$$\leq \exp \left[\frac{\left(M - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - m \right)}{M - m} \right.$$

$$\left. \times \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

$$\leq \exp \left[\frac{1}{4}(M - m) \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]$$

for each $x \in H$ with $\|x\| = 1$.

Remark 125 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$(1 \leq) \frac{m^{\frac{\langle Ax, x \rangle - M}{M - m}} M^{\frac{m - \langle Ax, x \rangle}{M - m}}}{\exp \langle \ln A^{-1} x, x \rangle} \qquad (2.216)$$

$$\leq \exp \left[\frac{\langle (M 1_H - A)(A - m 1_H) x, x \rangle}{Mm} \right]$$

$$\leq \exp \left[\frac{(M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)}{Mm} \right]$$

$$\leq \exp \left[\frac{1}{4} \frac{(M - m)^2}{mM} \right]$$

for all $x \in H$ with $\|x\| = 1$.

2.7 Hermite-Hadamard's Type Inequalities

2.7.1 Scalar Case

If $f : I \to \mathbb{R}$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f \left(\frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2}. \qquad (HH)$$

This remarkable result is well known in the literature as the *Hermite-Hadamard inequality* [29].

For various generalizations, extensions, reverses and related inequalities, see [1], [2], [19], [21], [24], [25], [27], [29] the monograph [18] and the references therein.

2.7.2 Some Inequalities for Convex Functions

The following inequality related to the Mond-Pečarić result also holds:

Theorem 126 (Dragomir, 2010, [14]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$.*

If f is a convex function on $[m, M]$, then

$$\frac{f(m) + f(M)}{2} \geq \left\langle \frac{f(A) + f((m + M)\,1_H - A)}{2} x, x \right\rangle \tag{2.217}$$

$$\geq \frac{f(\langle Ax, x \rangle) + f(m + M - \langle Ax, x \rangle)}{2}$$

$$\geq f\left(\frac{m + M}{2}\right)$$

for each $x \in H$ with $\|x\| = 1$.

In addition, if $x \in H$ with $\|x\| = 1$ and $\langle Ax, x \rangle \neq \frac{m+M}{2}$, then also

$$\frac{f(\langle Ax, x \rangle) + f(m + M - \langle Ax, x \rangle)}{2} \tag{2.218}$$

$$\geq \frac{2}{\frac{m+M}{2} - \langle Ax, x \rangle} \int_{\langle Ax, x \rangle}^{m+M-\langle Ax, x \rangle} f(u)\, du \geq f\left(\frac{m + M}{2}\right).$$

Proof. Since f is convex on $[m, M]$ then for each $u \in [m, M]$ we have the inequalities

$$\frac{M - u}{M - m} f(m) + \frac{u - m}{M - m} f(M) \tag{2.219}$$

$$\geq f\left(\frac{M - u}{M - m} m + \frac{u - m}{M - m} M\right) = f(u)$$

and

$$\frac{M - u}{M - m} f(M) + \frac{u - m}{M - m} f(m) \tag{2.220}$$

$$\geq f\left(\frac{M - u}{M - m} M + \frac{u - m}{M - m} m\right)$$

$$= f(M + m - u).$$

If we add these two inequalities we get

$$f(m) + f(M) \geq f(u) + f(M + m - u)$$

for any $u \in [m, M]$, which, by the property (P) applied for the operator A, produces the first inequality in (2.217).

By the Mond-Pečarić inequality we have

$$\langle f((m + M) 1_H - A) x, x \rangle \geq f(m + M - \langle Ax, x \rangle),$$

which together with the same inequality produces the second inequality in (2.217).

The third part follows by the convexity of f.

In order to prove (2.218), we use the Hermite-Hadamard inequality (HH) for the convex functions f and the choices $a = \langle Ax, x \rangle$ and $b = m + M - \langle Ax, x \rangle$.

The proof is complete. ∎

Remark 127 *We observe that, from the inequality (2.217) we have the following inequality in the operator order of $B(H)$*

$$\left[\frac{f(m) + f(M)}{2}\right] 1_H \geq \frac{f(A) + f((m + M) 1_H - A)}{2} \qquad (2.221)$$

$$\geq f\left(\frac{m + M}{2}\right) 1_H,$$

where f is a convex function on $[m, M]$ and A a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$.

The case of log-convex functions may be of interest for applications and therefore is stated in:

Corollary 128 (Dragomir, 2010, [14]) *If g is a log-convex function on $[m, M]$, then*

$$\sqrt{g(m) g(M)} \geq \exp \left\langle \ln [g(A) g((m + M) 1_H - A)]^{1/2} x, x \right\rangle \qquad (2.222)$$

$$\geq \sqrt{g(\langle Ax, x \rangle) g(m + M - \langle Ax, x \rangle)}$$

$$\geq g\left(\frac{m + M}{2}\right)$$

for each $x \in H$ with $\|x\| = 1$.

In addition, if $x \in H$ with $\|x\| = 1$ and $\langle Ax, x \rangle \neq \frac{m+M}{2}$, then also

$$\sqrt{g(\langle Ax, x \rangle) g(m + M - \langle Ax, x \rangle)} \qquad (2.223)$$

$$\geq \exp \left[\frac{2}{\frac{m+M}{2} - \langle Ax, x \rangle} \int_{\langle Ax, x \rangle}^{m+M-\langle Ax, x \rangle} \ln g(u) \, du\right]$$

$$\geq g\left(\frac{m + M}{2}\right).$$

The following result also holds

Theorem 129 (Dragomir, 2010, [14]) *Let A and B selfadjoint operators on the Hilbert space H and assume that $Sp\,(A)\,,Sp\,(B) \subseteq [m, M]$ for some scalars m, M with $m < M$.*

If f is a convex function on $[m, M]$, then

$$f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \qquad\qquad (2.224)$$

$$\leq \frac{1}{2}\left[f\left((1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle\right) + f\left(t\langle Ax, x\rangle + (1-t)\langle Bx, x\rangle\right)\right]$$

$$\leq \left\langle \frac{1}{2}\left[f\left((1-t)A + tB\right) + f\left(tA + (1-t)B\right)\right]x, x \right\rangle$$

$$\leq \frac{M - \left\langle \frac{A+B}{2}x, x\right\rangle}{M - m}f(m) + \frac{\left\langle \frac{A+B}{2}x, x\right\rangle - m}{M - m}f(M)$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1$.

Moreover, we have the Hermite-Hadamard's type inequalities:

$$f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \qquad\qquad (2.225)$$

$$\leq \int_0^1 f\left((1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle\right) dt$$

$$\leq \left\langle \left[\int_0^1 f\left((1-t)A + tB\right) dt\right]x, x \right\rangle$$

$$\leq \frac{M - \left\langle \frac{A+B}{2}x, x\right\rangle}{M - m}f(m) + \frac{\left\langle \frac{A+B}{2}x, x\right\rangle - m}{M - m}f(M)$$

each $x \in H$ with $\|x\| = 1$.

In addition, if we assume that $B - A$ is a positive definite operator, then

$$f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right)\langle (B-A)x, x\rangle \qquad\qquad (2.226)$$

$$\leq \int_{\langle Ax,x\rangle}^{\langle Bx,x\rangle} f(u)\, du \leq \langle (B-A)x, x\rangle \left\langle \left[\int_0^1 f\left((1-t)A + tB\right) dt\right]x, x \right\rangle$$

$$\leq \langle (B-A)x, x\rangle \left[\frac{M - \left\langle \frac{A+B}{2}x, x\right\rangle}{M - m}f(m) + \frac{\left\langle \frac{A+B}{2}x, x\right\rangle - m}{M - m}f(M)\right].$$

Proof. It is obvious that for any $t \in [0, 1]$ we have

$$Sp\left((1-t)A + tB\right), Sp\left(tA + (1-t)B\right) \subseteq [m, M].$$

On making use of the Mond-Pečarić inequality we have

$$f\left((1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle\right) \leq \langle f\left((1-t)A + tB\right)x, x\rangle \qquad (2.227)$$

and

$$f\left(t\left\langle Ax,x\right\rangle +\left(1-t\right)\left\langle Bx,x\right\rangle \right)\leq \left\langle f\left(tA+\left(1-t\right)B\right)x,x\right\rangle \qquad (2.228)$$

for any $t\in [0,1]$ and each $x\in H$ with $\|x\|=1$.

Adding (2.227) with (2.228) and utilising the convexity of f we deduce the first two inequalities in (2.224).

By the Lah-Ribarić inequality (2.143) we also have

$$\left\langle f\left(\left(1-t\right)A+tB\right)x,x\right\rangle \qquad (2.229)$$
$$\leq \frac{M-\left(1-t\right)\left\langle Ax,x\right\rangle -t\left\langle Bx,x\right\rangle }{M-m}\cdot f\left(m\right)$$
$$+\frac{\left(1-t\right)\left\langle Ax,x\right\rangle +t\left\langle Bx,x\right\rangle -m}{M-m}\cdot f\left(M\right)$$

and

$$\left\langle f\left(tA+\left(1-t\right)B\right)x,x\right\rangle \qquad (2.230)$$
$$\leq \frac{M-t\left\langle Ax,x\right\rangle -\left(1-t\right)\left\langle Bx,x\right\rangle }{M-m}\cdot f\left(m\right)$$
$$+\frac{t\left\langle Ax,x\right\rangle +\left(1-t\right)\left\langle Bx,x\right\rangle -m}{M-m}\cdot f\left(M\right)$$

for any $t\in [0,1]$ and each $x\in H$ with $\|x\|=1$.

Now, if we add the inequalities (2.229) with (2.230) and divide by two, we deduce the last part in (2.224).

Integrating the inequality over $t\in [0,1]$, utilising the continuity property of the inner product and the properties of the integral of operator-valued functions we have

$$f\left(\left\langle \frac{A+B}{2}x,x\right\rangle \right) \qquad (2.231)$$
$$\leq \frac{1}{2}\left[\int_{0}^{1}f\left(\left(1-t\right)\left\langle Ax,x\right\rangle +t\left\langle Bx,x\right\rangle \right)dt\right.$$
$$+\left.\int_{0}^{1}f\left(t\left\langle Ax,x\right\rangle +\left(1-t\right)\left\langle Bx,x\right\rangle \right)dt\right]$$
$$\leq \left\langle \frac{1}{2}\left[\int_{0}^{1}f\left(\left(1-t\right)A+tB\right)dt+\int_{0}^{1}f\left(tA+\left(1-t\right)B\right)dt\right]x,x\right\rangle$$
$$\leq \frac{M-\left\langle \frac{A+B}{2}x,x\right\rangle }{M-m}f\left(m\right)+\frac{\left\langle \frac{A+B}{2}x,x\right\rangle -m}{M-m}f\left(M\right).$$

Since

$$\int_{0}^{1}f\left(\left(1-t\right)\left\langle Ax,x\right\rangle +t\left\langle Bx,x\right\rangle \right)dt=\int_{0}^{1}f\left(t\left\langle Ax,x\right\rangle +\left(1-t\right)\left\langle Bx,x\right\rangle \right)dt$$

and

$$\int_0^1 f\left((1-t)A + tB\right) dt = \int_0^1 f\left(tA + (1-t)B\right) dt$$

then, by (2.231), we deduce the inequality (2.225).

The inequality (2.226) follows from (2.225) by observing that for $\langle Bx, x \rangle > \langle Ax, x \rangle$ we have

$$\int_0^1 f\left((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle\right) dt = \frac{1}{\langle Bx, x \rangle - \langle Ax, x \rangle} \int_{\langle Ax,x \rangle}^{\langle Bx,x \rangle} f(u)\, du$$

for each $x \in H$ with $\|x\| = 1$. ∎

Remark 130 *We observe that, from the inequalities (2.224) and (2.225) we have the following inequalities in the operator order of $B(H)$*

$$\frac{1}{2}\left[f\left((1-t)A + tB\right) + f\left(tA + (1-t)B\right)\right] \tag{2.232}$$

$$\leq f(m)\frac{M1_H - \frac{A+B}{2}}{M-m} + f(M)\frac{\frac{A+B}{2} - m1_H}{M-m},$$

where f is a convex function on $[m, M]$ and A, B are selfadjoint operator on the Hilbert space H with $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars m, M with $m < M$.

The case of log-convex functions is as follows:

Corollary 131 (Dragomir, 2010, [14]) *If g is a log-convex function on $[m, M]$, then*

$$g\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \tag{2.233}$$

$$\leq \sqrt{g\left((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle\right) g\left(t\langle Ax, x \rangle + (1-t)\langle Bx, x \rangle\right)}$$

$$\leq \exp\left\langle \frac{1}{2}\left[\ln g\left((1-t)A + tB\right) + \ln g\left(tA + (1-t)B\right)\right]x, x \right\rangle$$

$$\leq g(m)^{\frac{M - \langle \frac{A+B}{2}x,x \rangle}{M-m}} g(M)^{\frac{\langle \frac{A+B}{2}x,x \rangle - m}{M-m}}$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1$.

Moreover, we have the Hermite-Hadamard's type inequalities:

$$g\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \tag{2.234}$$

$$\leq \exp\left[\int_0^1 \ln g\left((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle\right) dt\right]$$

$$\leq \exp\left\langle \left[\int_0^1 \ln g\left((1-t)A + tB\right) dt\right]x, x \right\rangle$$

$$\leq g(m)^{\frac{M - \langle \frac{A+B}{2}x,x \rangle}{M-m}} g(M)^{\frac{\langle \frac{A+B}{2}x,x \rangle - m}{M-m}}$$

for each $x \in H$ with $\|x\| = 1$.

In addition, if we assume that $B - A$ is a positive definite operator, then

$$g\left(\left\langle \frac{A+B}{2}x, x\right\rangle\right)^{\langle(B-A)x,x\rangle} \tag{2.235}$$

$$\leq \exp\left[\int_{\langle Ax,x\rangle}^{\langle Bx,x\rangle} \ln g\left(u\right) du\right]$$

$$\leq \exp\left[\langle(B-A)x,x\rangle\left\langle\left[\int_0^1 \ln g\left((1-t)A+tB\right) dt\right]x, x\right\rangle\right]$$

$$\leq \left[g\left(m\right)^{\frac{M-\left\langle\frac{A+B}{2}x,x\right\rangle}{M-m}} g\left(M\right)^{\frac{\left\langle\frac{A+B}{2}x,x\right\rangle-m}{M-m}}\right]^{\langle(B-A)x,x\rangle}$$

for each $x \in H$ with $\|x\| = 1$.

From a different perspective we have the following result as well:

Theorem 132 (Dragomir, 2010, [14]) *Let A and B selfadjoint opera-tors on the Hilbert space H and assume that $Sp\left(A\right), Sp\left(B\right) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f\left(\frac{\langle Ax, x\rangle + \langle By, y\rangle}{2}\right) \tag{2.236}$$

$$\leq \int_0^1 f\left((1-t)\langle Ax, x\rangle + t\langle By, y\rangle\right) dt$$

$$\leq \left\langle\left[\int_0^1 f\left((1-t)A + t\langle By, y\rangle 1_H\right) dt\right]x, x\right\rangle$$

$$\leq \frac{1}{2}\left[\langle f\left(A\right)x, x\rangle + f\left(\langle By, y\rangle\right)\right]$$

$$\leq \frac{1}{2}\left[\langle f\left(A\right)x, x\rangle + \langle f\left(B\right)y, y\rangle\right]$$

and

$$f\left(\frac{\langle Ax, x\rangle + \langle By, y\rangle}{2}\right) \leq \left\langle f\left(\frac{A + \langle By, y\rangle 1_H}{2}\right)x, x\right\rangle \tag{2.237}$$

$$\leq \left\langle\left[\int_0^1 f\left((1-t)A + t\langle By, y\rangle 1_H\right) dt\right]x, x\right\rangle$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. For a convex function f and any $u, v \in [m, M]$ and $t \in [0, 1]$ we have the double inequality:

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\left[f\left((1-t)u+tv\right)+f\left(tu+(1-t)v\right)\right] \qquad (2.238)$$

$$\leq \frac{1}{2}\left[f(u)+f(v)\right].$$

Utilising the second inequality in (2.238) we have

$$\frac{1}{2}\left[f\left((1-t)u+t\langle By, y\rangle\right)+f\left(tu+(1-t)\langle By, y\rangle\right)\right] \qquad (2.239)$$

$$\leq \frac{1}{2}\left[f(u)+f(\langle By, y\rangle)\right]$$

for any $u \in [m, M]$, $t \in [0, 1]$ and $y \in H$ with $\|y\| = 1$.

Now, on applying the property (P) to the inequality (2.239) for the operator A we have

$$\frac{1}{2}\left[\langle f\left((1-t)A+t\langle By, y\rangle\right)x, x\rangle+\langle f\left(tA+(1-t)\langle By, y\rangle\right)x, x\rangle\right] \quad (2.240)$$

$$\leq \frac{1}{2}\left[\langle f(A)x, x\rangle+f(\langle By, y\rangle)\right]$$

for any $t \in [0, 1]$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.

On applying the Mond-Pečarić inequality we also have

$$\frac{1}{2}\left[f\left((1-t)\langle Ax, x\rangle+t\langle By, y\rangle\right)+f\left(t\langle Ax, x\rangle+(1-t)\langle By, y\rangle\right)\right] \quad (2.241)$$

$$\leq \frac{1}{2}\left[\langle f\left((1-t)A+t\langle By, y\rangle 1_H\right)x, x\rangle+\langle f\left(tA+(1-t)\langle By, y\rangle 1_H\right)x, x\rangle\right]$$

for any $t \in [0, 1]$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, integrating over t on $[0, 1]$ the inequalities (2.240) and (2.241) and taking into account that

$$\int_0^1 \langle f\left((1-t)A+t\langle By, y\rangle 1_H\right)x, x\rangle\, dt$$

$$= \int_0^1 \langle f\left(tA+(1-t)\langle By, y\rangle 1_H\right)x, x\rangle\, dt$$

$$= \left\langle \left[\int_0^1 f\left((1-t)A+t\langle By, y\rangle 1_H\right)dt\right]x, x\right\rangle$$

and

$$\int_0^1 f\left((1-t)\langle Ax, x\rangle+t\langle By, y\rangle\right)dt = \int_0^1 f\left(t\langle Ax, x\rangle+(1-t)\langle By, y\rangle\right)dt,$$

we obtain the second and the third inequality in (2.236).

Further, on applying the Jensen integral inequality for the convex function f we also have

$$\int_0^1 f\left((1-t)\langle Ax, x\rangle + t\langle By, y\rangle\right) dt$$

$$\geq f\left(\int_0^1 \left[(1-t)\langle Ax, x\rangle + t\langle By, y\rangle\right] dt\right)$$

$$= f\left(\frac{\langle Ax, x\rangle + \langle By, y\rangle}{2}\right)$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, proving the first part of (2.236).

Now, on utilising the first part of (2.238) we can also state that

$$f\left(\frac{u + \langle By, y\rangle}{2}\right) \tag{2.242}$$

$$\leq \frac{1}{2}\left[f\left((1-t)u + t\langle By, y\rangle\right) + f\left(tu + (1-t)\langle By, y\rangle\right)\right]$$

for any $u \in [m, M]$, $t \in [0, 1]$ and $y \in H$ with $\|y\| = 1$.

Further, on applying the property (P) to the inequality (2.242) and for the operator A we get

$$\left\langle f\left(\frac{A + \langle By, y\rangle 1_H}{2}\right) x, x\right\rangle$$

$$\leq \frac{1}{2}\left[\langle f\left((1-t)A + t\langle By, y\rangle 1_H\right)x, x\rangle + \langle f\left(tA + (1-t)\langle By, y\rangle 1_H\right)x, x\rangle\right]$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, which, by integration over t in $[0, 1]$ produces the second inequality in (2.237). The first inequality is obvious. ∎

Remark 133 *It is important to remark that, from the inequalities (2.236) and (2.237) we have the following Hermite-Hadamard's type results in the operator order of $B(H)$ and for the convex function $f : [m, M] \to \mathbb{R}$*

$$f\left(\frac{A + \langle By, y\rangle 1_H}{2}\right) \leq \int_0^1 f\left((1-t)A + t\langle By, y\rangle 1_H\right) dt \tag{2.243}$$

$$\leq \frac{1}{2}\left[f(A) + f\left(\langle By, y\rangle\right) 1_H\right]$$

for any $y \in H$ with $\|y\| = 1$ and any selfadjoint operators A, B with spectra in $[m, M]$.

In particular, we have from (2.243)

$$f\left(\frac{A + \langle Ay, y\rangle 1_H}{2}\right) \leq \int_0^1 f\left((1-t)A + t\langle Ay, y\rangle 1_H\right) dt \tag{2.244}$$

$$\leq \frac{1}{2}\left[f(A) + f\left(\langle Ay, y\rangle\right) 1_H\right]$$

for any $y \in H$ with $\|y\| = 1$ and

$$f\left(\frac{A + s1_H}{2}\right) \leq \int_0^1 f\left((1 - t)A + ts1_H\right) dt \tag{2.245}$$

$$\leq \frac{1}{2}\left[f(A) + f(s)1_H\right]$$

for any $s \in [m, M]$.

As a particular case of the above theorem we have the following refinement of the Mond-Pečarić inequality:

Corollary 134 (Dragomir, 2010, [14]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f\left(\langle Ax, x\rangle\right) \leq \left\langle f\left(\frac{A + \langle Ax, x\rangle 1_H}{2}\right) x, x\right\rangle \tag{2.246}$$

$$\leq \left\langle \left[\int_0^1 f\left((1 - t)A + t\langle Ax, x\rangle 1_H\right) dt\right] x, x\right\rangle$$

$$\leq \frac{1}{2}\left[\langle f(A)x, x\rangle + f\left(\langle Ax, x\rangle\right)\right] \leq \langle f(A)x, x\rangle.$$

Finally, the case of log-convex functions is as follows:

Corollary 135 (Dragomir, 2010, [14]) *If g is a log-convex function on $[m, M]$, then*

$$g\left(\frac{\langle Ax, x\rangle + \langle By, y\rangle}{2}\right) \tag{2.247}$$

$$\leq \exp\left[\int_0^1 \ln g\left((1 - t)\langle Ax, x\rangle + t\langle By, y\rangle\right) dt\right]$$

$$\leq \exp\left\langle \left[\int_0^1 \ln g\left((1 - t)A + t\langle By, y\rangle 1_H\right) dt\right] x, x\right\rangle$$

$$\leq \exp\left[\frac{1}{2}\left[\langle \ln g(A)x, x\rangle + \ln g\left(\langle By, y\rangle\right)\right]\right]$$

$$\leq \exp\left[\frac{1}{2}\left[\langle \ln g(A)x, x\rangle + \langle \ln g(B)y, y\rangle\right]\right]$$

and

$$g\left(\frac{\langle Ax, x\rangle + \langle By, y\rangle}{2}\right) \tag{2.248}$$

$$\leq \exp\left\langle \ln g\left(\frac{A + \langle By, y\rangle 1_H}{2}\right) x, x\right\rangle$$

$$\leq \exp\left\langle \left[\int_0^1 \ln g\left((1 - t)A + t\langle By, y\rangle 1_H\right) dt\right] x, x\right\rangle$$

and

$$g\left(\langle Ax, x \rangle\right) \tag{2.249}$$

$$\leq \exp\left\langle \ln g\left(\frac{A + \langle Ax, x \rangle 1_H}{2}\right) x, x \right\rangle$$

$$\leq \exp\left\langle \left[\int_0^1 \ln g\left((1-t)A + t\langle Ax, x \rangle 1_H\right) dt\right] x, x \right\rangle$$

$$\leq \exp\left[\frac{1}{2}\left[\langle \ln g\left(A\right) x, x \rangle + \ln g\left(\langle Ax, x \rangle\right)\right]\right] \leq \exp\langle \ln g\left(A\right) x, x \rangle$$

respectively, for each $x \in H$ *with* $\|x\| = 1$ *and* A, B *selfadjoint operators with spectra in* $[m, M]$.

It is obvious that all the above inequalities can be applied for particular convex or log-convex functions of interest. However, we will restrict ourselves to only a few examples that are connected with famous results such as the Hölder-McCarthy inequality or the Ky Fan inequality.

2.7.3   Applications for Hölder-McCarthy's Inequality

We can improve the Hölder-McCarthy's inequality above as follows:

Proposition 136 *Let* A *be a selfadjoint positive operator on a Hilbert space* H.

If $r > 1$, *then*

$$\langle Ax, x \rangle^r \leq \left\langle \left(\frac{A + \langle Ax, x \rangle 1_H}{2}\right)^r x, x \right\rangle \tag{2.250}$$

$$\leq \left\langle \left[\int_0^1 \left((1-t)A + t\langle Ax, x \rangle 1_H\right)^r dt\right] x, x \right\rangle$$

$$\leq \frac{1}{2}\left[\langle A^r x, x \rangle + \langle Ax, x \rangle^r\right] \leq \langle A^r x, x \rangle$$

for any $x \in H$ *with* $\|x\| = 1$.

If $0 < r < 1$, *then the inequalities reverse in (2.250).*

If A *is invertible and* $r > 0$, *then*

$$\langle Ax, x \rangle^{-r} \leq \left\langle \left(\frac{A + \langle Ax, x \rangle 1_H}{2}\right)^{-r} x, x \right\rangle \tag{2.251}$$

$$\leq \left\langle \left[\int_0^1 \left((1-t)A + t\langle Ax, x \rangle 1_H\right)^{-r} dt\right] x, x \right\rangle$$

$$\leq \frac{1}{2}\left[\langle A^{-r} x, x \rangle + \langle Ax, x \rangle^{-r}\right] \leq \langle A^{-r} x, x \rangle$$

for any $x \in H$ *with* $\|x\| = 1$.

Follows from the inequality (2.247) applied for the power function.

Since the function $g(t) = t^{-r}$ for $r > 0$ is log-convex, then by utilising the inequality (2.249) we can improve the Hölder-McCarthy inequality as follows:

Proposition 137 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$\langle Ax, x \rangle^{-r} \leq \exp \left\langle \ln \left(\frac{A + \langle Ax, x \rangle 1_H}{2} \right)^{-r} x, x \right\rangle \tag{2.252}$$

$$\leq \exp \left\langle \left[\int_0^1 \ln \left((1 - t) A + t \langle Ax, x \rangle 1_H \right)^{-r} dt \right] x, x \right\rangle$$

$$\leq \exp \left[\frac{1}{2} \left[\langle \ln A^{-r} x, x \rangle + \ln \langle Ax, x \rangle^{-r} \right] \right] \leq \exp \langle \ln A^{-r} x, x \rangle$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

Now, from a different perspective, we can state the following operator power inequalities:

Proposition 138 *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M] \subset [0, \infty)$, then*

$$\frac{m^r + M^r}{2} \geq \left\langle \frac{A^r + ((m + M) 1_H - A)^r}{2} x, x \right\rangle \tag{2.253}$$

$$\geq \frac{\langle Ax, x \rangle^r + (m + M - \langle Ax, x \rangle)^r}{2} \geq \left(\frac{m + M}{2} \right)^r$$

for each $x \in H$ with $\|x\| = 1$ and $r > 1$.
If $0 < r < 1$ then the inequalities reverse in (2.253).
If A is positive definite and $r > 0$, then

$$\frac{m^{-r} + M^{-r}}{2} \geq \left\langle \frac{A^{-r} + ((m + M) 1_H - A)^{-r}}{2} x, x \right\rangle \tag{2.254}$$

$$\geq \frac{\langle Ax, x \rangle^{-r} + (m + M - \langle Ax, x \rangle)^{-r}}{2} \geq \left(\frac{m + M}{2} \right)^{-r}$$

for each $x \in H$ with $\|x\| = 1$.

The proof follows by the inequality (2.217).

Finally we have:

Proposition 139 *Assume that A and B are selfadjoint operators with spectra in $[m, M] \subset [0, \infty)$ and $x \in H$ with $\|x\| = 1$ and such that $\langle Ax, x \rangle \neq \langle Bx, x \rangle$.*

If $r > 1$ or $r \in (\infty, -1) \cup (-1, 0)$ then we have

$$\left\langle \left(\frac{A+B}{2}\right) x, x \right\rangle^r \leq \frac{1}{r+1} \cdot \frac{\langle Ax, x\rangle^{r+1} - \langle Bx, x\rangle^{r+1}}{\langle Ax, x\rangle - \langle Bx, x\rangle} \tag{2.255}$$

$$\leq \left\langle \left[\int_0^1 ((1-t)A + tB)^r \, dt \right] x, x \right\rangle$$

$$\leq \frac{M - \left\langle \frac{A+B}{2} x, x \right\rangle}{M - m} m^r + \frac{\left\langle \frac{A+B}{2} x, x \right\rangle - m}{M - m} M^r.$$

If $0 < r < 1$, then the inequalities reverse in (2.255).
If A and B are positive definite, then

$$\left\langle \left(\frac{A+B}{2}\right) x, x \right\rangle^{-1} \leq \frac{\ln \langle Bx, x\rangle - \ln \langle Ax, x\rangle}{\langle Bx, x\rangle - \langle Ax, x\rangle} \tag{2.256}$$

$$\leq \left\langle \left[\int_0^1 ((1-t)A + tB)^{-1} \, dt \right] x, x \right\rangle$$

$$\leq \frac{M - \left\langle \frac{A+B}{2} x, x \right\rangle}{(M - m) m} + \frac{\left\langle \frac{A+B}{2} x, x \right\rangle - m}{(M - m) M}.$$

2.7.4 Applications for Ky Fan's Inequality

The following results related to the Ky Fan inequality may be stated as well:

Proposition 140 *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subset \left(0, \frac{1}{2}\right)$, then*

$$\left(\langle (1_H - A) x, x \rangle \langle Ax, x \rangle^{-1}\right)^r \tag{2.257}$$

$$\leq \exp \left\langle \ln \left([1_H - A + \langle (1_H - A) x, x \rangle 1_H] (A + \langle Ax, x \rangle 1_H)^{-1}\right)^r x, x \right\rangle$$

$$\leq \left\langle \exp \left[\int_0^1 [\ln ((1-t)(1_H - A) + t \langle (1_H - A) x, x \rangle 1_H) \right.\right.$$

$$\left.\left. \times ((1-t)A + t \langle Ax, x \rangle 1_H)^{-1}]^r \, dt \right] x, x \right\rangle$$

$$\leq \exp \left[\frac{1}{2} \left[\left\langle \ln \left[(1_H - A) A^{-1}\right]^r x, x \right\rangle + \ln \left(\langle (1_H - A) x, x \rangle \langle Ax, x \rangle^{-1}\right)^r\right] \right]$$

$$\leq \exp \left\langle \ln \left[(1_H - A) A^{-1}\right]^r x, x \right\rangle$$

for any $x \in H$ with $\|x\| = 1$.

It follows from the inequality (2.249) applied for the log-convex function $g : (0, 1) \to \mathbb{R}$, $g(t) = \left(\frac{1-t}{t}\right)^r$, $r > 0$.

Proposition 141 *Assume that A is a selfadjoint operator with $Sp\,(A) \subset \left(0, \frac{1}{2}\right)$ and $s \in \left(0, \frac{1}{2}\right)$. Then we have the following inequality in the operator order of $B\,(H)$:*

$$\ln\left[[(2-s)\,1_H - A]\,(A + s1_H)^{-1}\right] \tag{2.258}$$

$$\leq \int_0^1 \ln\left([(1-ts)\,1_H - (1-t)\,A]\,((1-t)\,A + ts1_H)^{-1}\right)dt$$

$$\leq \frac{1}{2}\left(\ln\left[(1_H - A)\,A^{-1}\right]^r + \ln\left(\frac{1-s}{s}\right)^r 1_H\right).$$

If follows from the inequality (2.245) applied for the log-convex function $g : (0,1) \to \mathbb{R}$, $g\,(t) = \left(\frac{1-t}{t}\right)^r, r > 0$.

2.8 Hermite-Hadamard's Type Inequalities for Operator Convex Functions

2.8.1 Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx \tag{2.259}$$

$$< (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}.$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [29]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [36].

E.F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [3]. In 1974, D.S. Mitrinović found Hermite's note in *Mathesis* [29]. Since (2.259) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [36].

Let X be a vector space, $x, y \in X$, $x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty, \ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0,1] \to \mathbb{R}, \ g(x, y)(t) := f[(1-t)x + ty], \ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x.y] \subset X$, we have the Hermite-Hadamard integral inequality (see [4, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x)+f(y)}{2}, \tag{2.260}$$

which can be derived from the classical Hermite-Hadamard inequality (2.259) for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, we have the following norm inequality from (2.260) (see [35, p. 106])

$$\left\| \frac{x+y}{2} \right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}, \qquad (2.261)$$

for any $x, y \in X$.

Motivated by the above results we investigate in this paper the operator version of the Hermite-Hadamard inequality for operator convex functions. The operator quasilinearity of some associated functionals are also provided.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B) \qquad (OC)$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $Sp(A), Sp(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [20] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

2.8.2 Some Hermite-Hadamard's Type Inequalities

We start with the following result:

Theorem 142 (Dragomir, 2010, [13]) *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A*

and B with spectra in I we have the inequality

$$\left(f\left(\frac{A+B}{2}\right)\leq\right)\frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right] \qquad (2.262)$$

$$\leq \int_0^1 f\left((1-t)A+tB\right)dt$$

$$\leq\frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f\left(A\right)+f\left(B\right)}{2}\right]\left(\leq\frac{f\left(A\right)+f\left(B\right)}{2}\right).$$

Proof. First of all, since the function f is continuos, the operator valued integral $\int_0^1 f\left((1-t)A+tB\right)dt$ exists for any selfadjoint operators A and B with spectra in I.

We give here two proofs, the first using only the definition of operator convex functions and the second using the classical Hermite-Hadamard inequality for real valued functions.

1. By the definition of operator convex functions we have the double inequality:

$$f\left(\frac{C+D}{2}\right)\leq\frac{1}{2}\left[f\left((1-t)C+tD\right)+f\left((1-t)D+tC\right)\right] \qquad (2.263)$$

$$\leq\frac{1}{2}\left[f\left(C\right)+f\left(D\right)\right]$$

for any $t\in[0,1]$ and any selfadjoint operators C and D with the spectra in I.

Integrating the inequality (2.263) over $t\in[0,1]$ and taking into account that

$$\int_0^1 f\left((1-t)C+tD\right)dt=\int_0^1 f\left((1-t)D+tC\right)dt$$

then we deduce the Hermite-Hadamard inequality for operator convex functions

$$f\left(\frac{C+D}{2}\right)\leq\int_0^1 f\left((1-t)C+tD\right)dt \qquad (\text{HHO})$$

$$\leq\frac{1}{2}\left[f\left(C\right)+f\left(D\right)\right]$$

that holds for any selfadjoint operators C and D with the spectra in I.

Now, on making use of the change of variable $u=2t$ we have

$$\int_0^{1/2} f\left((1-t)A+tB\right)dt=\frac{1}{2}\int_0^1 f\left((1-u)A+u\frac{A+B}{2}\right)du$$

and by the change of variable $u=2t-1$ we have

$$\int_{1/2}^1 f\left((1-t)A+tB\right)dt=\frac{1}{2}\int_0^1 f\left((1-u)\frac{A+B}{2}+uB\right)du.$$

Utilising the Hermite-Hadamard inequality (HHO) we can write

$$f\left(\frac{3A+B}{4}\right) \le \int_0^1 f\left((1-u)A + u\frac{A+B}{2}\right) du$$
$$\le \frac{1}{2}\left[f(A) + f\left(\frac{A+B}{2}\right)\right]$$

and

$$f\left(\frac{A+3B}{4}\right) \le \int_0^1 f\left((1-u)\frac{A+B}{2} + uB\right) du$$
$$\le \frac{1}{2}\left[f(A) + f\left(\frac{A+B}{2}\right)\right],$$

which by summation and division by two produces the desired result (2.262).

2. Consider now $x \in H, \|x\| = 1$ and two selfadjoint operators A and B with spectra in I. Define the real-valued function $\varphi_{x,A,B} : [0,1] \to \mathbb{R}$ given by $\varphi_{x,A,B}(t) = \langle f((1-t)A + tB)x, x\rangle$.

Since f is operator convex, then for any $t_1, t_2 \in [0,1]$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ we have

$$\varphi_{x,A,B}(\alpha t_1 + \beta t_2)$$
$$= \langle f((1-(\alpha t_1 + \beta t_2))A + (\alpha t_1 + \beta t_2)B)x, x\rangle$$
$$= \langle f(\alpha[(1-t_1)A + t_1 B] + \beta[(1-t_2)A + t_2 B])x, x\rangle$$
$$\le \alpha \langle f([(1-t_1)A + t_1 B])x, x\rangle + \beta \langle f([(1-t_2)A + t_2 B])x, x\rangle$$
$$= \alpha\varphi_{x,A,B}(t_1) + \beta\varphi_{x,A,B}(t_2)$$

showing that $\varphi_{x,A,B}$ is a convex function on $[0,1]$.

Now we use the Hermite-Hadamard inequality for real-valued convex functions

$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b g(s)\,ds \le \frac{g(a)+g(b)}{2}$$

to get that

$$\varphi_{x,A,B}\left(\frac{1}{4}\right) \le 2\int_0^{1/2} \varphi_{x,A,B}(t)\,dt \le \frac{\varphi_{x,A,B}(0) + \varphi_{x,A,B}\left(\frac{1}{2}\right)}{2}$$

and

$$\varphi_{x,A,B}\left(\frac{3}{4}\right) \le 2\int_{1/2}^1 \varphi_{x,A,B}(t)\,dt \le \frac{\varphi_{x,A,B}\left(\frac{1}{2}\right) + \varphi_{x,A,B}(1)}{2}$$

which by summation and division by two produces

$$\frac{1}{2} \left\langle \left[f\left(\frac{3A+B}{4} \right) + f\left(\frac{A+3B}{4} \right) \right] x, x \right\rangle \qquad (2.264)$$

$$\leq \int_0^1 \langle f\left((1-t) A + tB \right) x, x \rangle \, dt$$

$$\leq \frac{1}{2} \left\langle \left[f\left(\frac{A+B}{2} \right) + \frac{f\left(A \right) + f\left(B \right)}{2} \right] x, x \right\rangle.$$

Finally, since by the continuity of the function f we have

$$\int_0^1 \langle f\left((1-t) A + tB \right) x, x \rangle \, dt = \left\langle \int_0^1 f\left((1-t) A + tB \right) dt x, x \right\rangle$$

for any $x \in H, \|x\| = 1$ and any two selfadjoint operators A and B with spectra in I, we deduce from (2.264) the desired result (2.262). $\blacksquare$

A simple consequence of the above theorem is that the integral is closer to the left bound than to the right, namely we can state:

Corollary 143 (Dragomir, 2010, [13]) *With the assumptions in Theorem 142 we have the inequality*

$$(0 \leq) \int_0^1 f\left((1-t) A + tB \right) dt - f\left(\frac{A+B}{2} \right) \qquad (2.265)$$

$$\leq \frac{f\left(A \right) + f\left(B \right)}{2} - \int_0^1 f\left((1-t) A + tB \right) dt.$$

Remark 144 *Utilising different examples of operator convex or concave functions, we can provide inequalities of interest.*

If $r \in [-1, 0] \cup [1, 2]$ then we have the inequalities for powers of operators

$$\left(\left(\frac{A+B}{2} \right)^r \leq \right) \frac{1}{2} \left[\left(\frac{3A+B}{4} \right)^r + \left(\frac{A+3B}{4} \right)^r \right] \qquad (2.266)$$

$$\leq \int_0^1 \left((1-t) A + tB \right)^r dt$$

$$\leq \frac{1}{2} \left[\left(\frac{A+B}{2} \right)^r + \frac{A^r + B^r}{2} \right] \left(\leq \frac{A^r + B^r}{2} \right)$$

for any two selfadjoint operators A and B with spectra in $(0, \infty)$.

If $r \in (0, 1)$ the inequalities in (2.266) hold with " $\geq$ " instead of " $\leq$ ".

We also have the following inequalities for logarithm

$$\left(\ln \left(\frac{A+B}{2} \right) \geq \right) \frac{1}{2} \left[\ln \left(\frac{3A+B}{4} \right) + \ln \left(\frac{A+3B}{4} \right) \right] \qquad (2.267)$$

$$\geq \int_0^1 \ln \left((1-t) A + tB \right) dt$$

$$\geq \frac{1}{2} \left[\ln \left(\frac{A+B}{2} \right) + \frac{\ln (A) + \ln (B)}{2} \right] \left(\geq \frac{\ln (A) + \ln (B)}{2} \right)$$

for any two selfadjoint operators A and B with spectra in $(0, \infty)$.

2.8.3   Some Operator Quasilinearity Properties

Consider an operator convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ defined on the interval I and two distinct selfadjoint operators A, B with the spectra in I. We denote by $[A, B]$ the closed operator segment defined by the family of operators $\{(1 - t) A + tB, \, t \in [0, 1]\}$. We also define the operator-valued functional

$$\Delta_f (A, B; t) := (1 - t) f (A) + t f (B) - f ((1 - t) A + tB) \geq 0 \qquad (2.268)$$

in the operator order, for any $t \in [0, 1]$.

The following result concerning an operator quasilinearity property for the functional $\Delta_f (\cdot, \cdot; t)$ may be stated:

Theorem 145 (Dragomir, 2010, [13]) *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I. Then for each A, B two distinct selfadjoint operators A, B with the spectra in I and $C \in [A, B]$ we have*

$$(0 \leq) \Delta_f (A, C; t) + \Delta_f (C, B; t) \leq \Delta_f (A, B; t) \qquad (2.269)$$

for each $t \in [0, 1]$, i.e., the functional $\Delta_f (\cdot, \cdot; t)$ is operator superadditive as a function of interval.

If $[C, D] \subset [A, B]$, then

$$(0 \leq) \Delta_f (C, D; t) \leq \Delta_f (A, B; t) \qquad (2.270)$$

for each $t \in [0, 1]$, i.e., the functional $\Delta_f (\cdot, \cdot; t)$ is operator nondecreasing as a function of interval.

Proof. Let $C = (1 - s) A + sB$ with $s \in (0, 1)$. For $t \in (0, 1)$ we have

$$\Delta_f (C, B; t) = (1 - t) f ((1 - s) A + sB) + t f (B)$$
$$- f ((1 - t) [(1 - s) A + sB] + tB)$$

and

$$\Delta_f (A, C; t) = (1 - t) f (A) + t f ((1 - s) A + sB)$$
$$- f ((1 - t) A + t [(1 - s) A + sB])$$

giving that

$$\Delta_f \left(A, C; t \right) + \Delta_f \left(C, B; t \right) - \Delta_f \left(A, B; t \right) \tag{2.271}$$
$$= f \left((1 - s) A + sB \right) + f \left((1 - t) A + tB \right)$$
$$- f \left((1 - t) (1 - s) A + \left[(1 - t) s + t \right] B \right) - f \left((1 - ts) A + tsB \right).$$

Now, for a convex function $\varphi : I \subset \mathbb{R} \rightarrow \mathbb{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \le s_1$ and $t_2 \le s_2$ we have that

$$\frac{\varphi \left(t_1 \right) - \varphi \left(t_2 \right)}{t_1 - t_2} \le \frac{\varphi \left(s_1 \right) - \varphi \left(s_2 \right)}{s_1 - s_2}. \tag{2.272}$$

Indeed, since φ is convex on I then for any $a \in I$ the function $\psi : I \backslash \{a\} \rightarrow \mathbb{R}$

$$\psi \left(t \right) := \frac{\varphi \left(t \right) - \varphi \left(a \right)}{t - a}$$

is monotonic nondecreasing where is defined. Utilising this property repeatedly we have

$$\begin{aligned}
\frac{\varphi \left(t_1 \right) - \varphi \left(t_2 \right)}{t_1 - t_2} &\le \frac{\varphi \left(s_1 \right) - \varphi \left(t_2 \right)}{s_1 - t_2} = \frac{\varphi \left(t_2 \right) - \varphi \left(s_1 \right)}{t_2 - s_1} \\
&\le \frac{\varphi \left(s_2 \right) - \varphi \left(s_1 \right)}{s_2 - s_1} = \frac{\varphi \left(s_1 \right) - \varphi \left(s_2 \right)}{s_1 - s_2}
\end{aligned}$$

which proves the inequality (2.272).

For a vector $x \in H$, with $\|x\| = 1$, consider the function $\varphi_x : [0, 1] \rightarrow \mathbb{R}$ given by $\varphi_x \left(t \right) := \langle f \left((1 - t) A + tB \right) x, x \rangle$. Since f is operator convex on I it follows that φ_x is convex on $[0, 1]$. Now, if we consider, for given $t, s \in (0, 1)$,

$$t_1 := ts < s =: s_1 \text{ and } t_2 := t < t + (1 - t) s =: s_2,$$

then we have

$$\varphi_x \left(t_1 \right) = \langle f \left((1 - ts) A + tsB \right) x, x \rangle$$

and

$$\varphi_x \left(t_2 \right) = \langle f \left((1 - t) A + tB \right) x, x \rangle$$

giving that

$$\frac{\varphi_x \left(t_1 \right) - \varphi_x \left(t_2 \right)}{t_1 - t_2} = \left\langle \left[\frac{f \left((1 - ts) A + tsB \right) - f \left((1 - t) A + tB \right)}{t \left(s - 1 \right)} \right] x, x \right\rangle.$$

Also

$$\varphi_x \left(s_1 \right) = \langle f \left((1 - s) A + sB \right) x, x \rangle$$

and

$$\varphi_x \left(s_2 \right) = \langle f \left((1 - t) (1 - s) A + \left[(1 - t) s + t \right] B \right) x, x \rangle$$

giving that

$$\frac{\varphi_x\left(s_1\right) - \varphi_x\left(s_2\right)}{s_1 - s_2}$$
$$= \left\langle \frac{f\left(\left(1-s\right)A + sB\right) - f\left(\left(1-t\right)\left(1-s\right)A + \left[\left(1-t\right)s + t\right]B\right)}{t\left(s-1\right)} x, x \right\rangle.$$

Utilising the inequality (2.272) and multiplying with $t\left(s-1\right) < 0$ we deduce the following inequality in the operator order

$$f\left(\left(1-ts\right)A + tsB\right) - f\left(\left(1-t\right)A + tB\right) \tag{2.273}$$
$$\geq f\left(\left(1-s\right)A + sB\right) - f\left(\left(1-t\right)\left(1-s\right)A + \left[\left(1-t\right)s + t\right]B\right).$$

Finally, by (2.271) and (2.273) we get the desired result (2.269).

Applying repeatedly the superadditivity property we have for $[C, D] \subset [A, B]$ that

$$\Delta_f\left(A, C; t\right) + \Delta_f\left(C, D; t\right) + \Delta_f\left(D, B; t\right) \leq \Delta_f\left(A, B; t\right)$$

giving that

$$0 \leq \Delta_f\left(A, C; t\right) + \Delta_f\left(D, B; t\right) \leq \Delta_f\left(A, B; t\right) - \Delta_f\left(C, D; t\right)$$

which proves (2.270). ∎

For $t = \frac{1}{2}$ we consider the functional

$$\Delta_f\left(A, B\right) := \Delta_f\left(A, B; \frac{1}{2}\right) = \frac{f\left(A\right) + f\left(B\right)}{2} - f\left(\frac{A+B}{2}\right),$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Delta_f\left(\cdot, \cdot; t\right)$. We are able then to state the following

Corollary 146 (Dragomir, 2010, [13]) *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be an operator convex function on the interval* I. *Then for each* A, B *two distinct selfadjoint operators* A, B *with the spectra in* I *we have the following bounds in the operator order*

$$\inf_{C\in[A,B]} \left[f\left(\frac{A+C}{2}\right) + f\left(\frac{C+B}{2}\right) - f\left(C\right) \right] = f\left(\frac{A+B}{2}\right) \tag{2.274}$$

and

$$\sup_{C,D\in[A,B]} \left[\frac{f\left(C\right) + f\left(D\right)}{2} - f\left(\frac{C+D}{2}\right) \right] \tag{2.275}$$
$$= \frac{f\left(A\right) + f\left(B\right)}{2} - f\left(\frac{A+B}{2}\right).$$

Proof. By the superadditivity of the functional $\Delta_f(\cdot, \cdot)$ we have for each $C \in [A, B]$ that

$$\frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2}\right)$$

$$\geq \frac{f(A) + f(C)}{2} - f\left(\frac{A+C}{2}\right) + \frac{f(C) + f(B)}{2} - f\left(\frac{C+B}{2}\right)$$

which is equivalent with

$$f\left(\frac{A+C}{2}\right) + f\left(\frac{C+B}{2}\right) - f(C) \geq f\left(\frac{A+B}{2}\right). \qquad (2.276)$$

Since the equality case in (2.276) is realized for either $C = A$ or $C = B$ we get the desired bound (2.274).

The bound (2.275) is obvious by the monotonicity of the functional $\Delta_f(\cdot, \cdot)$ as a function of interval. ∎

Consider now the following functional

$$\Gamma_f(A, B; t) := f(A) + f(B) - f((1-t)A + tB) - f((1-t)B + tA),$$

where, as above, $f : C \subset X \to \mathbb{R}$ is a convex function on the convex set C and $A, B \in C$ while $t \in [0, 1]$.

We notice that

$$\Gamma_f(A, B; t) = \Gamma_f(B, A; t) = \Gamma_f(A, B; 1 - t)$$

and

$$\Gamma_f(A, B; t) = \Delta_f(A, B; t) + \Delta_f(A, B; 1 - t) \geq 0$$

for any $A, B \in C$ and $t \in [0, 1]$.

Therefore, we can state the following result as well

Corollary 147 (Dragomir, 2010, [13]) *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I. Then for each A, B two distinct selfadjoint operators A, B with the spectra in I, the functional $\Gamma_f(\cdot, \cdot; t)$ is operator superadditive and operator nondecreasing as a function of interval.*

In particular, if $C \in [A, B]$ then we have the inequality

$$\frac{1}{2}[f((1-t)A + tB) + f((1-t)B + tA)] \qquad (2.277)$$

$$\leq \frac{1}{2}[f((1-t)A + tC) + f((1-t)C + tA)]$$

$$+ \frac{1}{2}[f((1-t)C + tB) + f((1-t)B + tC)] - f(C).$$

Also, if $C, D \in [A, B]$ then we have the inequality

$$f(A) + f(B) - f((1-t)A + tB) - f((1-t)B + tA) \quad (2.278)$$
$$\geq f(C) + f(D) - f((1-t)C + tD) - f((1-t)C + tD)$$

for any $t \in [0, 1]$.

Perhaps the most interesting functional we can consider is the following one:

$$\Theta_f(A, B) = \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB) \, dt. \quad (2.279)$$

Notice that, by the second Hermite-Hadamard inequality for operator convex functions we have that $\Theta_f(A, B) \geq 0$ in the operator order.

We also observe that

$$\Theta_f(A, B) = \int_0^1 \Delta_f(A, B; t) \, dt = \int_0^1 \Delta_f(A, B; 1-t) \, dt. \quad (2.280)$$

Utilising this representation, we can state the following result as well:

Corollary 148 (Dragomir, 2010, [13]) *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I. Then for each A, B two distinct selfadjoint operators A, B with the spectra in I, the functional $\Theta_f(\cdot, \cdot)$ is operator superadditive and operator nondecreasing as a function of interval. Moreover, we have the bounds in the operator order*

$$\inf_{C \in [A,B]} \left[\int_0^1 [f((1-t)A + tC) + f((1-t)C + tB)] \, dt - f(C) \right] \quad (2.281)$$
$$= \int_0^1 f((1-t)A + tB) \, dt$$

and

$$\sup_{C,D \in [A,B]} \left[\frac{f(C) + f(D)}{2} - \int_0^1 f((1-t)C + tD) \, dt \right] \quad (2.282)$$
$$= \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB) \, dt.$$

Remark 149 *The above inequalities can be applied to various concrete operator convex function of interest.*

If we choose for instance the inequality (2.282), then we get the following bounds in the operator order

$$\sup_{C,D \in [A,B]} \left[\frac{C^r + D^r}{2} - \int_0^1 ((1-t)C + tD)^r \, dt \right] \quad (2.283)$$
$$= \frac{A^r + B^r}{2} - \int_0^1 ((1-t)A + tB)^r \, dt,$$

where $r \in [-1, 0] \cup [1, 2]$ *and* A, B *are selfadjoint operators with spectra in* $(0, \infty)$.

If $r \in (0, 1)$ *then*

$$\sup_{C, D \in [A, B]} \left[\int_0^1 ((1 - t) C + tD)^r \, dt - \frac{C^r + D^r}{2} \right] \qquad (2.284)$$

$$= \int_0^1 ((1 - t) A + tB)^r \, dt - \frac{A^r + B^r}{2},$$

and A, B *are selfadjoint operators with spectra in* $(0, \infty)$.

We also have the operator bound for the logarithm

$$\sup_{C, D \in [A, B]} \left[\int_0^1 \ln((1 - t) C + tD) \, dt - \frac{\ln(C) + \ln(D)}{2} \right] \qquad (2.285)$$

$$= \int_0^1 \ln((1 - t) A + tB) \, dt - \frac{\ln(A) + \ln(B)}{2},$$

where A, B *are selfadjoint operators with spectra in* $(0, \infty)$.

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3
Inequalities for the Čebyšev Functional

3.1 Introduction

The *Čebyšev*, or in a different spelling, *Chebyshev inequality* which compares the integral/discrete mean of the product with the product of the integral/discrete means is famous in the literature devoted to Mathematical Inequalities. It has been extended, generalised, refined etc...by many authors during the last century. A simple search utilising either spellings and the key word "inequality" in the title in the comprehensive *MathSciNet* database of the *American Mathematical Society* produces more than 200 research articles devoted to this result.

The sister result due to Grüss which provides error bounds for the magnitude of the difference between the integral mean of the product and the product of the integral means has also attracted much interest since it has been discovered in 1935 with more than 180 papers published, as a simple search in the same database reveals. Far more publications have been devoted to the applications of these inequalities and an accurate picture of the impacted results in various fields of Modern Mathematics is difficult to provide.

In this chapter, however, we present only some recent results due to the author for the corresponding operator versions of these two famous inequalities. Applications for particular functions of selfadjoint operators such as the power, logarithmic and exponential functions are provided as well.

3.2   Čebyšev's Inequality

3.2.1   Čebyšev's Inequality for Real Numbers

First of all, let us recall a number of classical results for sequences of real numbers concerning the celebrated *Čebyšev inequality*.

Consider the real sequences $(n-\text{tuples})$ $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ and the nonnegative sequence $\mathbf{p} = (p_1, \ldots, p_n)$ with $P_n := \sum_{i=1}^{n} p_i > 0$. Define the *weighted Čebyšev's functional*

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) := \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i. \qquad (3.1)$$

In $1882-1883$, Čebyšev [7] and [8] proved that if $\mathbf{a}$ and $\mathbf{b}$ are *monotonic* in the same (opposite) sense, then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq (\leq) 0. \qquad (3.2)$$

In the special case $\mathbf{p} = \mathbf{a} \geq \mathbf{0}$, it appears that the inequality (3.2) has been obtained by Laplace long before Čebyšev (see for example [51, p. 240]).

The inequality (3.2) was mentioned by Hardy, Littlewood and Pólya in their book [46] in 1934 in the more general setting of synchronous sequences, i.e., if $\mathbf{a}$, $\mathbf{b}$ are *synchronous* (*asynchronous*), this means that

$$(a_i - a_j)(b_i - b_j) \geq (\leq) 0 \text{ for any } i, j \in \{1, \ldots, n\}, \qquad (3.3)$$

then (3.2) holds true as well.

A relaxation of the synchronicity condition was provided by M. Biernacki in 1951, [5], which showed that, if $\mathbf{a}$, $\mathbf{b}$ are *monotonic in mean* in the same sense, i.e., for $P_k := \sum_{i=1}^{k} p_i$, $k = 1, \ldots, n-1$;

$$\frac{1}{P_k} \sum_{i=1}^{k} p_i a_i \leq (\geq) \frac{1}{P_{k+1}} \sum_{i=1}^{k+1} p_i a_i, \quad k \in \{1, \ldots, n-1\} \qquad (3.4)$$

and

$$\frac{1}{P_k} \sum_{i=1}^{k} p_i b_i \leq (\geq) \frac{1}{P_{k+1}} \sum_{i=1}^{k+1} p_i b_i, \quad k \in \{1, \ldots, n-1\}, \qquad (3.5)$$

then (3.2) holds with " $\geq$ ". If if $\mathbf{a}$, $\mathbf{b}$ are monotonic in mean in the opposite sense then (3.2) holds with " $\leq$ ".

If one would like to drop the assumption of nonnegativity for the components of $\mathbf{p}$, then one may state the following inequality obtained by Mitrinović and Pečarić in 1991, [50]: If $0 \leq P_i \leq P_n$ for each $i \in \{1, \ldots, n-1\}$, then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq 0 \qquad (3.6)$$

provided **a** and **b** are sequences with the same monotonicity.

If **a** and **b** are monotonic in the opposite sense, the sign of the inequality (3.6) reverses.

Similar integral inequalities may be stated, however we do not present them here.

For other recent results on the Čebyšev inequality in either discrete or integral form see [6], [19], [20], [26], [39], [40], [51], [49], [52], [57], [58], [59], and the references therein.

The main aim of the present section is to provide operator versions for the Čebyšev inequality in different settings. Related results and some particular cases of interest are also given.

3.2.2   A Version of the Čebyšev Inequality for One Operator

We say that the functions $f, g : [a, b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0 \quad \text{for each } t, s \in [a, b].$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

For some extensions of the discrete *Čebyšev inequality* for *synchronous (asynchronous)* sequences of vectors in an inner product space, see [42] and [41].

The following result provides an inequality of Čebyšev type for functions of selfadjoint operators.

Theorem 150 (Dragomir, 2008, [30]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\langle f(A) g(A) x, x \rangle \geq (\leq) \langle f(A) x, x \rangle \cdot \langle g(A) x, x \rangle \qquad (3.7)$$

for any $x \in H$ with $\|x\| = 1$.

Proof. We consider only the case of synchronous functions. In this case we have then

$$f(t) g(t) + f(s) g(s) \geq f(t) g(s) + f(s) g(t) \qquad (3.8)$$

for each $t, s \in [a, b]$.

If we fix $s \in [a, b]$ and apply the property (P) for the inequality (3.8) then we have for each $x \in H$ with $\|x\| = 1$ that

$$\langle (f(A) g(A) + f(s) g(s) 1_H) x, x \rangle \geq \langle (g(s) f(A) + f(s) g(A)) x, x \rangle,$$

which is clearly equivalent with

$$\langle f(A) g(A) x, x \rangle + f(s) g(s) \geq g(s) \langle f(A) x, x \rangle + f(s) \langle g(A) x, x \rangle \quad (3.9)$$

for each $s \in [a, b]$.

Now, if we apply again the property (P) for the inequality (3.9), then we have for any $y \in H$ with $\|y\| = 1$ that

$$\langle ((\langle f(A) g(A) x, x \rangle 1_H + f(A) g(A)) y, y \rangle$$
$$\geq \langle ((\langle f(A) x, x \rangle g(A) + \langle g(A) x, x \rangle f(A)) y, y \rangle,$$

which is clearly equivalent with

$$\langle f(A) g(A) x, x \rangle + \langle f(A) g(A) y, y \rangle \qquad (3.10)$$
$$\geq \langle f(A) x, x \rangle \langle g(A) y, y \rangle + \langle f(A) y, y \rangle \langle g(A) x, x \rangle$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. This is an inequality of interest in itself.

Finally, on making $y = x$ in (3.10) we deduce the desired result (3.7). ∎

Some particular cases are of interest for applications. In the first instance we consider the case of power functions.

Example 151 *Assume that A is a positive operator on the Hilbert space H and $p, q > 0$. Then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\langle A^{p+q} x, x \rangle \geq \langle A^p x, x \rangle \cdot \langle A^q x, x \rangle. \qquad (3.11)$$

If A is positive definite then the inequality (3.11) also holds for $p, q < 0$.

If A is positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then the reverse inequality holds in (3.11).

Another case of interest for applications is the exponential function.

Example 152 *Assume that A is a selfadjoint operator on H. If $\alpha, \beta > 0$ or $\alpha, \beta < 0$, then*

$$\langle \exp[(\alpha + \beta) A] x, x \rangle \geq \langle \exp(\alpha A) x, x \rangle \cdot \langle \exp(\beta A) x, x \rangle \qquad (3.12)$$

for each $x \in H$ with $\|x\| = 1$.

If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then the reverse inequality holds in (3.12).

The following particular cases may be of interest as well:

Example 153 a. *Assume that A is positive definite and $p > 0$. Then*

$$\langle A^p \log A x, x \rangle \geq \langle A^p x, x \rangle \cdot \langle \log A x, x \rangle \qquad (3.13)$$

for each $x \in H$ with $\|x\| = 1$. If $p < 0$ then the reverse inequality holds in (3.13).

b. *Assume that A is positive definite and $Sp\,(A) \subset (0, 1)$. If $r, s > 0$ or $r, s < 0$ then*

$$\left\langle (1_H - A^r)^{-1} (1_H - A^s)^{-1} x, x \right\rangle \tag{3.14}$$
$$\geq \left\langle (1_H - A^r)^{-1} x, x \right\rangle \cdot \left\langle (1_H - A^s)^{-1} x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$.

If either $r > 0, s < 0$ or $r < 0, s > 0$, then the reverse inequality holds in (3.14).

Remark 154 *We observe, from the proof of the above theorem that, if A and B are selfadjoint operators and $Sp\,(A)\,, Sp\,(B) \subseteq [m, M]\,,$ then for any continuous synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the more general result*

$$\langle f\,(A)\,g\,(A)\,x, x \rangle + \langle f\,(B)\,g\,(B)\,y, y \rangle \tag{3.15}$$
$$\geq (\leq) \langle f\,(A)\,x, x \rangle \langle g\,(B)\,y, y \rangle + \langle f\,(B)\,y, y \rangle \langle g\,(A)\,x, x \rangle$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

If $f : [m, M] \longrightarrow (0, \infty)$ is continuous then the functions f^p, f^q are synchronous in the case when $p, q > 0$ or $p, q < 0$ and asynchronous when either $p > 0, q < 0$ or $p < 0, q > 0$. In this situation if A and B are positive definite operators then we have the inequality

$$\left\langle f^{p+q}\,(A)\,x, x \right\rangle + \left\langle f^{p+q}\,(B)\,y, y \right\rangle \tag{3.16}$$
$$\geq \langle f^p\,(A)\,x, x \rangle \langle f^q\,(B)\,y, y \rangle + \langle f^p\,(B)\,y, y \rangle \langle f^q\,(A)\,x, x \rangle$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$ where either $p, q > 0$ or $p, q < 0$. If $p > 0, q < 0$ or $p < 0, q > 0$ then the reverse inequality also holds in (3.16).

As particular cases, we should observe that for $p = q = 1$ and $f\,(t) = t$, we get from (3.16) the inequality

$$\left\langle A^2 x, x \right\rangle + \left\langle B^2 y, y \right\rangle \geq 2 \cdot \langle Ax, x \rangle \langle By, y \rangle \tag{3.17}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

For $p = 1$ and $q = -1$ we have from (3.16)

$$\langle Ax, x \rangle \left\langle B^{-1} y, y \right\rangle + \langle By, y \rangle \left\langle A^{-1} x, x \right\rangle \leq 2 \tag{3.18}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

3.2.3   A Version of the Čebyšev Inequality for n Operators

The following multiple operator version of Theorem 150 holds:

Theorem 155 (Dragomir, 2008, [30]) *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle \tag{3.19}$$

$$\geq (\leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle,$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. As in [44, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M], \|\widetilde{x}\| = 1,$

$$\left\langle f\left(\widetilde{A}\right) g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle,$$

$$\left\langle f\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \text{ and } \left\langle g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle.$$

Applying Theorem 150 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result (3.19). ∎
    The following particular cases may be of interest for applications.

Example 156 *Assume that $A_j, j \in \{1, \ldots, n\}$ are positive operators on the Hilbert space H and $p, q > 0$. Then for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$ we have the inequality*

$$\left\langle \sum_{j=1}^{n} A_j^{p+q} x_j, x_j \right\rangle \geq \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle. \tag{3.20}$$

If A_j are positive definite then the inequality (3.20) also holds for $p, q < 0$.
    If A_j are positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then the reverse inequality holds in (3.20).

    Another case of interest for applications is the exponential function.

Example 157 *Assume that $A_j, j \in \{1, \ldots, n\}$ are selfadjoint operators on H. If $\alpha, \beta > 0$ or $\alpha, \beta < 0$, then*

$$\left\langle \sum_{j=1}^{n} \exp\left[(\alpha + \beta) A_j\right] x_j, x_j \right\rangle \tag{3.21}$$

$$\geq \sum_{j=1}^{n} \left\langle \exp\left(\alpha A_j\right) x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \exp\left(\beta A_j\right) x_j, x_j \right\rangle$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.
   If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then the reverse inequality holds in (3.21).

The following particular cases may be of interest as well:

Example 158 *a.* *Assume that $A_j, j \in \{1, \ldots, n\}$ are positive definite operators and $p > 0$. Then*

$$\left\langle \sum_{j=1}^{n} A_j^p \log A_j x_j, x_j \right\rangle \geq \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \log A_j x_j, x_j \right\rangle \tag{3.22}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. If $p < 0$ then the reverse inequality holds in (3.22).
   b. *If A_j are positive definite and $Sp(A_j) \subset (0,1)$ for $j \in \{1, \ldots, n\}$ then for $r, s > 0$ or $r, s < 0$ we have the inequality*

$$\left\langle \sum_{j=1}^{n} \left(1_H - A_j^r\right)^{-1} \left(1_H - A_j^s\right)^{-1} x_j, x_j \right\rangle \tag{3.23}$$

$$\geq \sum_{j=1}^{n} \left\langle \left(1_H - A_j^r\right)^{-1} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \left(1_H - A_j^s\right)^{-1} x_j, x_j \right\rangle$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.
   If either $r > 0, s < 0$ or $r < 0, s > 0$, then the reverse inequality holds in (3.23).

3.2.4   Another Version of the Čebyšev Inequality for n Operators

The following different version of the Čebyšev inequality for a sequence of operators also holds:

Theorem 159 (Dragomir, 2008, [30]) *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$.*

If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \tag{3.24}$$

$$\geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle,$$

for any $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$. In particular

$$\left\langle \frac{1}{n} \sum_{j=1}^{n} f(A_j) g(A_j) x, x \right\rangle \tag{3.25}$$

$$\geq (\leq) \left\langle \frac{1}{n} \sum_{j=1}^{n} f(A_j) x, x \right\rangle \cdot \left\langle \frac{1}{n} \sum_{j=1}^{n} g(A_j) x, x \right\rangle,$$

for each $x \in H$ with $\|x\| = 1$.

Proof. We provide here two proofs. The first is based on the inequality (3.15) and generates as a by-product a more general result. The second is derived from Theorem 155.

1. If we make use of the inequality (3.15), then we can write

$$\langle f(A_j) g(A_j) x, x \rangle + \langle f(B_k) g(B_k) y, y \rangle \tag{3.26}$$
$$\geq (\leq) \langle f(A_j) x, x \rangle \langle g(B_k) y, y \rangle + \langle f(B_k) y, y \rangle \langle g(A_j) x, x \rangle,$$

which holds for any A_j and B_k selfadjoint operators with $Sp(A_j), Sp(B_k) \subseteq [m, M], j, k \in \{1, \ldots, n\}$ and for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, if $p_j \geq 0, q_k \geq 0, j, k \in \{1, \ldots, n\}$ and $\sum_{j=1}^{n} p_j = \sum_{k=1}^{n} q_k = 1$ then, by multiplying (3.26) with $p_j \geq 0, q_k \geq 0$ and summing over j and k from 1 to n we deduce the following inequality that is of interest in its own right:

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle + \left\langle \sum_{k=1}^{n} q_k f(B_k) g(B_k) y, y \right\rangle \tag{3.27}$$

$$\geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \left\langle \sum_{k=1}^{n} q_k g(B_k) y, y \right\rangle$$

$$+ \left\langle \sum_{k=1}^{n} q_k f(B_k) y, y \right\rangle \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, the choice $B_k = A_k, q_k = p_k$ and $y = x$ in (3.27) produces the desired result (3.24).

2. In we choose in Theorem 155 $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$ then a simple calculation shows that the inequality (3.19) becomes (3.24). The details are omitted. ∎

Remark 160 *We remark that the case $n = 1$ in (3.24) produces the inequality (3.7).*

The following particular cases are of interest:

Example 161 *Assume that $A_j, j \in \{1, \ldots, n\}$ are positive operators on the Hilbert space H, $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $p, q > 0$. Then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle \geq \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle. \qquad (3.28)$$

If $A_j, j \in \{1, \ldots, n\}$ are positive definite then the inequality (3.28) also holds for $p, q < 0$.

If $A_j, j \in \{1, \ldots, n\}$ are positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then the reverse inequality holds in (3.28).

Another case of interest for applications is the exponential function.

Example 162 *Assume that $A_j, j \in \{1, \ldots, n\}$ are selfadjoint operators on H and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If $\alpha, \beta > 0$ or $\alpha, \beta < 0$, then*

$$\left\langle \sum_{j=1}^{n} p_j \exp\left[(\alpha + \beta) A_j\right] x, x \right\rangle \qquad (3.29)$$

$$\geq \left\langle \sum_{j=1}^{n} p_j \exp\left(\alpha A_j\right) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \exp\left(\beta A_j\right) x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$.

If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then the reverse inequality holds in (3.29).

The following particular cases may be of interest as well:

Example 163 *a. Assume that $A_j, j \in \{1, \ldots, n\}$ are positive definite operators on the Hilbert space H, $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$*

and $p > 0$. Then

$$\left\langle \sum_{j=1}^{n} p_j A_j^p \log A_j x, x \right\rangle \tag{3.30}$$

$$\geq \left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle.$$

If $p < 0$ then the reverse inequality holds in (3.30).

b. Assume that $A_j, j \in \{1, \ldots, n\}$ are positive definite operators on the Hilbert space $H, Sp(A_j) \subset (0, 1)$ and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If $r, s > 0$ or $r, s < 0$ then

$$\left\langle \sum_{j=1}^{n} p_j \left(1_H - A_j^r\right)^{-1} \left(1_H - A_j^s\right)^{-1} x, x \right\rangle \tag{3.31}$$

$$\geq \left\langle \sum_{j=1}^{n} p_j \left(1_H - A_j^r\right)^{-1} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \left(1_H - A_j^s\right)^{-1} x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$.

If either $r > 0, s < 0$ or $r < 0, s > 0$, then the reverse inequality holds in (3.31).

We remark that the following operator norm inequality can be stated as well:

Corollary 164 *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, asynchronous on $[m, M]$ and for $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ the operator $\sum_{j=1}^{n} p_j f(A_j) g(A_j)$ is positive, then*

$$\left\| \sum_{j=1}^{n} p_j f(A_j) g(A_j) \right\| \leq \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j g(A_j) \right\|. \tag{3.32}$$

Proof. We have from (3.24) that

$$0 \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$. Taking the supremum in this inequality over $x \in H$ with $\|x\| = 1$ we deduce the desired result (3.32). ∎

The above Corollary 164 provides some interesting norm inequalities for sums of positive operators as follows:

Example 165 *a. If $A_j, j \in \{1, \ldots, n\}$ are positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then for $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have the norm inequality:*

$$\left\| \sum_{j=1}^{n} p_j A_j^{p+q} \right\| \leq \left\| \sum_{j=1}^{n} p_j A_j^{p} \right\| \cdot \left\| \sum_{j=1}^{n} p_j A_j^{q} \right\|. \tag{3.33}$$

In particular

$$1 \leq \left\| \sum_{j=1}^{n} p_j A_j^{r} \right\| \cdot \left\| \sum_{j=1}^{n} p_j A_j^{-r} \right\| \tag{3.34}$$

for any $r > 0$.

b. Assume that $A_j, j \in \{1, \ldots, n\}$ are selfadjoint operators on H and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then

$$\left\| \sum_{j=1}^{n} p_j \exp\left[(\alpha + \beta) A_j \right] \right\| \tag{3.35}$$

$$\leq \left\| \sum_{j=1}^{n} p_j \exp\left(\alpha A_j \right) \right\| \cdot \left\| \sum_{j=1}^{n} p_j \exp\left(\beta A_j \right) \right\|.$$

In particular

$$1 \leq \left\| \sum_{j=1}^{n} p_j \exp\left(\gamma A_j \right) \right\| \cdot \left\| \sum_{j=1}^{n} p_j \exp\left(-\gamma A_j \right) \right\|.$$

for any $\gamma > 0$.

3.2.5 Related Results for One Operator

The following result that is related to the Čebyšev inequality may be stated:

Theorem 166 (Dragomir, 2008, [30]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m, M]$, then*

$$\langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \cdot \langle g(A) x, x \rangle \tag{3.36}$$
$$\geq \left[\langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \right] \cdot \left[g(\langle Ax, x \rangle) - \langle g(A) x, x \rangle \right]$$

for any $x \in H$ with $\|x\| = 1$.

If f, g are asynchronous, then

$$\langle f(A) x, x \rangle \cdot \langle g(A) x, x \rangle - \langle f(A) g(A) x, x \rangle \tag{3.37}$$
$$\geq \left[\langle f(A) x, x \rangle - f(\langle Ax, x \rangle) \right] \cdot \left[\langle g(A) x, x \rangle - g(\langle Ax, x \rangle) \right]$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f, g are synchronous and $m \leq \langle Ax, x \rangle \leq M$ for any $x \in H$ with $\|x\| = 1$, then we have

$$[f(t) - f(\langle Ax, x \rangle)][g(t) - g(\langle Ax, x \rangle)] \geq 0 \qquad (3.38)$$

for any $t \in [a, b]$ and $x \in H$ with $\|x\| = 1$.

On utilising the property (P) for the inequality (3.38) we have that

$$\langle [f(B) - f(\langle Ax, x \rangle)][g(B) - g(\langle Ax, x \rangle)] y, y \rangle \geq 0 \qquad (3.39)$$

for any B a bounded linear operator with $Sp(B) \subseteq [m, M]$ and $y \in H$ with $\|y\| = 1$.

Since

$$\begin{aligned}
&\langle [f(B) - f(\langle Ax, x \rangle)][g(B) - g(\langle Ax, x \rangle)] y, y \rangle \qquad (3.40) \\
&= \langle f(B) g(B) y, y \rangle + f(\langle Ax, x \rangle) g(\langle Ax, x \rangle) \\
&\quad - \langle f(B) y, y \rangle g(\langle Ax, x \rangle) - f(\langle Ax, x \rangle) \langle g(B) y, y \rangle,
\end{aligned}$$

then from (3.39) we get

$$\begin{aligned}
&\langle f(B) g(B) y, y \rangle + f(\langle Ax, x \rangle) g(\langle Ax, x \rangle) \\
&\geq \langle f(B) y, y \rangle g(\langle Ax, x \rangle) + f(\langle Ax, x \rangle) \langle g(B) y, y \rangle
\end{aligned}$$

which is clearly equivalent with

$$\begin{aligned}
&\langle f(B) g(B) y, y \rangle - \langle f(A) y, y \rangle \cdot \langle g(A) y, y \rangle \qquad (3.41) \\
&\geq [\langle f(B) y, y \rangle - f(\langle Ax, x \rangle)] \cdot [g(\langle Ax, x \rangle) - \langle g(B) y, y \rangle]
\end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. This inequality is of interest in its own right.

Now, if we choose $B = A$ and $y = x$ in (3.41), then we deduce the desired result (3.36). ∎

The following result which improves the Čebyšev inequality may be stated:

Corollary 167 (Dragomir, 2008, [30]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, synchronous and one is convex while the other is concave on $[m, M]$, then*

$$\begin{aligned}
&\langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \cdot \langle g(A) x, x \rangle \qquad (3.42) \\
&\geq [\langle f(A) x, x \rangle - f(\langle Ax, x \rangle)] \cdot [g(\langle Ax, x \rangle) - \langle g(A) x, x \rangle] \geq 0
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

If f, g are asynchronous and either both of them are convex or both of them concave on $[m, M]$, then

$$\begin{aligned}
&\langle f(A) x, x \rangle \cdot \langle g(A) x, x \rangle - \langle f(A) g(A) x, x \rangle \qquad (3.43) \\
&\geq [\langle f(A) x, x \rangle - f(\langle Ax, x \rangle)] \cdot [\langle g(A) x, x \rangle - g(\langle Ax, x \rangle)] \geq 0
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. The second inequality follows by making use of the result due to Mond & Pečarić, see [55], [54] or [44, p. 5]:

$$\langle h(A)x, x \rangle \geq (\leq) h(\langle Ax, x \rangle) \tag{MP}$$

for any $x \in H$ with $\|x\| = 1$ provided that A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and h is convex (concave) on the given interval $[m, M]$. ∎

The above Corollary 167 offers the possibility to improve some of the results established before for power function as follows:

Example 168 **a.** *Assume that A is a positive operator on the Hilbert space H. If $p \in (0, 1)$ and $q \in (1, \infty)$, then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\left\langle A^{p+q}x, x \right\rangle - \langle A^p x, x \rangle \cdot \langle A^q x, x \rangle \tag{3.44}$$
$$\geq [\langle A^q x, x \rangle - \langle Ax, x \rangle^q][\langle Ax, x \rangle^p - \langle A^p x, x \rangle] \geq 0.$$

If A is positive definite and $p > 1, q < 0$, then

$$\langle A^p x, x \rangle \cdot \langle A^q x, x \rangle - \left\langle A^{p+q}x, x \right\rangle \tag{3.45}$$
$$\geq [\langle A^q x, x \rangle - \langle Ax, x \rangle^q][\langle A^p x, x \rangle - \langle Ax, x \rangle^p] \geq 0$$

for each $x \in H$ with $\|x\| = 1$.

b. *Assume that A is positive definite and $p > 1$. Then*

$$\langle A^p \log Ax, x \rangle - \langle A^p x, x \rangle \cdot \langle \log Ax, x \rangle \tag{3.46}$$
$$\geq [\langle A^p x, x \rangle - \langle Ax, x \rangle^p][\log \langle Ax, x \rangle - \langle \log Ax, x \rangle] \geq 0$$

for each $x \in H$ with $\|x\| = 1$.

3.2.6   Related Results for n Operators

We can state now the following generalisation of Theorem 166 for n operators:

Theorem 169 (Dragomir, 2008, [30]) *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$.*

(i)   If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m, M]$, then

$$\sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle \tag{3.47}$$

$$\geq \left[\sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^n \langle A_j x_j, x_j \rangle \right) \right]$$

$$\times \left[g\left(\sum_{j=1}^n \langle A_j x_j, x_j \rangle \right) - \sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Moreover, if one function is convex while the other is concave on $[m, M]$, then the right hand side of (3.47) is nonnegative.

(ii) If f, g are asynchronous on $[m, M]$, then

$$\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle \quad (3.48)$$

$$\geq \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right]$$

$$\times \left[\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle - g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Moreover, if either both of them are convex or both of them are concave on $[m, M]$, then the right hand side of (3.48) is nonnegative as well.

Proof. The argument is similar to the one from the proof of Theorem 155 on utilising the results from one operator obtained in Theorem 166.

The nonnegativity of the right hand sides of the inequalities (3.47) and (3.48) follows by the use of the Jensen's type result from [44, p. 5]

$$\sum_{j=1}^{n} \langle h(A_j) x_j, x_j \rangle \geq (\leq) h\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \quad (3.49)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, which holds provided that A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$ and h is convex (concave) on $[m, M]$.

The details are omitted. ■

Example 170 a. Assume that $A_j, j \in \{1, \ldots, n\}$ are positive operators on the Hilbert space H. If $p \in (0, 1)$ and $q \in (1, \infty)$, then for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$ we have the inequality

$$\sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \quad (3.50)$$

$$\geq \left[\sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^q \right]$$

$$\times \left[\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^p - \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \right]$$

$$\geq 0.$$

If A_j are positive definite and $p > 1, q < 0$, then

$$\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^q x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle \qquad (3.51)$$

$$\geq \left[\sum_{j=1}^{n} \left\langle A_j^q x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right)^q \right]$$

$$\times \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right)^p \right]$$

$$\geq 0$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.
   b. *Assume that A_j are positive definite and $p > 1$. Then*

$$\sum_{j=1}^{n} \left\langle A_j^p \log A x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \log A_j x_j, x_j \right\rangle \qquad (3.52)$$

$$\geq \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right)^p \right]$$

$$\times \left[\sum_{j=1}^{n} \log \left\langle A_j x_j, x_j \right\rangle - \log \left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \right) \right]$$

$$\geq 0$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

The following result may be stated as well:

Theorem 171 (Dragomir, 2008, [30]) *Let A_j be selfadjoint operators with $Sp\,(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$.*

(i) *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous and synchronous on* $[m, M]$, *then*

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \tag{3.53}$$

$$- \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle$$

$$\geq \left[f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \right]$$

$$\times \left[\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle - g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \right]$$

for any $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$. *Moreover, if one is convex while the other is concave on* $[m, M]$, *then the right hand side of (3.53) is nonnegative.*

(ii) *If* f, g *are asynchronous on* $[m, M]$, *then*

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \tag{3.54}$$

$$- \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle$$

$$\geq \left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \right]$$

$$\times \left[\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle - g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \right]$$

for any $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$. *Moreover, if either both of them are convex or both of them are concave on* $[m, M]$, *then the right hand side of (3.54) is nonnegative as well.*

Proof. Follows from Theorem 169 on choosing $x_j = \sqrt{p_j} \cdot x, j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$.

Also, the positivity of the right hand term in (3.53) follows by the Jensen's type inequality from the inequality (3.49) for the same choices, namely $x_j = \sqrt{p_j} \cdot x, j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ■

Finally, we can list some particular inequalities that may be of interest for applications. They improve some result obtained above:

Example 172 a. *Assume that $A_j, j \in \{1, \ldots, n\}$ are positive operators on the Hilbert space H and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If $p \in (0,1)$ and $q \in (1, \infty)$, then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^q x, x \right\rangle \qquad (3.55)$$

$$\geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^q x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^q \right]$$

$$\times \left[\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^p - \left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \right]$$

$$\geq 0.$$

If $A_j, j \in \{1, \ldots, n\}$ are positive definite and $p > 1, q < 0$, then

$$\left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^q x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle \qquad (3.56)$$

$$\geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^q x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^q \right]$$

$$\times \left[\left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^p \right]$$

$$\geq 0$$

for each $x \in H$ with $\|x\| = 1$.
   b. *Assume that $A_j, j \in \{1, \ldots, n\}$ are positive definite and $p > 1$. Then*

$$\left\langle \sum_{j=1}^{n} p_j A_j^p \log A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle \qquad (3.57)$$

$$\geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^p \right]$$

$$\times \left[\log \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle \right]$$

$$\geq 0$$

for each $x \in H$ with $\|x\| = 1$.

3.3   Grüss Inequality

3.3.1   Some Elementary Inequalities of Grüss Type

In 1935, G. Grüss [45] proved the following integral inequality which gives
an approximation of the integral of the product in terms of the product of
the integrals as follows:

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \quad (3.58)$$

$$\leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma),$$

where $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi, \qquad \gamma \leq g(x) \leq \Gamma \quad (3.59)$$

for each $x \in [a, b]$, where $\phi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced
by a smaller one.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [51, Chapter X]
established the following discrete version of Grüss' inequality:

Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be two $n-$tuples of real numbers
such that $r \leq a_i \leq R$ and $s \leq b_i \leq S$ for $i = 1, \ldots, n$. Then one has

$$\left| \frac{1}{n} \sum_{i=1}^n a_i b_i - \frac{1}{n} \sum_{i=1}^n a_i \cdot \frac{1}{n} \sum_{i=1}^n b_i \right| \quad (3.60)$$

$$\leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (R - r)(S - s),$$

where $[x]$ denotes the integer part of x, $x \in \mathbb{R}$.

For a simple proof of (3.58) as well as for some other integral inequalities
of Grüss type, see Chapter X of the recent book [51]. For other related
results see the papers [1]-[4], [11]-[9], [12]-[13], [15]-[37], [43], [56], [62] and
the references therein.

3.3.2   An Inequality of Grüss' Type for One Operator

The following result may be stated:

Theorem 173 (Dragomir, 2008, [31]) *Let A be a selfadjoint operator
on the Hilbert space $(H; \langle ., . \rangle)$ and assume that $Sp(A) \subseteq [m, M]$ for some
scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$*

and $\Gamma := \max_{t \in [m,M]} f(t)$ *then*

$$|\langle f(A) g(A) y, y \rangle - \langle f(A) y, y \rangle \cdot \langle g(A) x, x \rangle \tag{3.61}$$
$$- \frac{\gamma + \Gamma}{2} [\langle g(A) y, y \rangle - \langle g(A) x, x \rangle]|$$
$$\leq \frac{1}{2} \cdot (\Gamma - \gamma) \left[\|g(A) y\|^2 + \langle g(A) x, x \rangle^2 - 2 \langle g(A) x, x \rangle \langle g(A) y, y \rangle \right]^{1/2}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1.$

Proof. First of all, observe that, for each $\lambda \in \mathbb{R}$ and $x, y \in H$, $\|x\| = \|y\| = 1$ we have the identity

$$\langle (f(A) - \lambda \cdot 1_H)(g(A) - \langle g(A) x, x \rangle \cdot 1_H) y, y \rangle \tag{3.62}$$
$$= \langle f(A) g(A) y, y \rangle - \lambda \cdot [\langle g(A) y, y \rangle - \langle g(A) x, x \rangle]$$
$$- \langle g(A) x, x \rangle \langle f(A) y, y \rangle.$$

Taking the modulus in (3.62) we have

$$|\langle f(A) g(A) y, y \rangle - \lambda \cdot [\langle g(A) y, y \rangle - \langle g(A) x, x \rangle] \tag{3.63}$$
$$- \langle g(A) x, x \rangle \langle f(A) y, y \rangle|$$
$$= |\langle (g(A) - \langle g(A) x, x \rangle \cdot 1_H) y, (f(A) - \lambda \cdot 1_H) y \rangle|$$
$$\leq \|g(A) y - \langle g(A) x, x \rangle y\| \|f(A) y - \lambda y\|$$
$$= \left[\|g(A) y\|^2 + \langle g(A) x, x \rangle^2 - 2 \langle g(A) x, x \rangle \langle g(A) y, y \rangle \right]^{1/2}$$
$$\times \|f(A) y - \lambda y\|$$
$$\leq \left[\|g(A) y\|^2 + \langle g(A) x, x \rangle^2 - 2 \langle g(A) x, x \rangle \langle g(A) y, y \rangle \right]^{1/2}$$
$$\times \|f(A) - \lambda \cdot 1_H\|$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, since $\gamma = \min_{t \in [m,M]} f(t)$ and $\Gamma = \max_{t \in [m,M]} f(t)$, then by the property (P) we have that $\gamma \leq \langle f(A) y, y \rangle \leq \Gamma$ for each $y \in H$ with $\|y\| = 1$ which is clearly equivalent with

$$\left| \langle f(A) y, y \rangle - \frac{\gamma + \Gamma}{2} \|y\|^2 \right| \leq \frac{1}{2} (\Gamma - \gamma)$$

or with

$$\left| \left\langle \left(f(A) - \frac{\gamma + \Gamma}{2} 1_H \right) y, y \right\rangle \right| \leq \frac{1}{2} (\Gamma - \gamma)$$

for each $y \in H$ with $\|y\| = 1$.

Taking the supremum in this inequality we get

$$\left\| f(A) - \frac{\gamma + \Gamma}{2} \cdot 1_H \right\| \leq \frac{1}{2} (\Gamma - \gamma),$$

which together with the inequality (3.63) applied for $\lambda = \frac{\gamma+\Gamma}{2}$ produces the desired result (3.61). ∎

As a particular case of interest we can derive from the above theorem the following result of Grüss' type:

Corollary 174 (Dragomir, 2008, [31]) *With the assumptions in Theorem 173 we have*

$$|\langle f(A)g(A)x, x\rangle - \langle f(A)x, x\rangle \cdot \langle g(A)x, x\rangle| \qquad (3.64)$$

$$\leq \frac{1}{2} \cdot (\Gamma - \gamma) \left[\|g(A)x\|^2 - \langle g(A)x, x\rangle^2\right]^{1/2}$$

$$\left(\leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta)\right)$$

for each $x \in H$ *with* $\|x\| = 1$, *where* $\delta := \min_{t \in [m,M]} g(t)$ *and* $\Delta := \max_{t \in [m,M]} g(t)$.

Proof. The first inequality follows from (3.61) by putting $y = x$.

Now, if we write the first inequality in (3.64) for $f = g$ we get

$$0 \leq \|g(A)x\|^2 - \langle g(A)x, x\rangle^2 = \langle g^2(A)x, x\rangle - \langle g(A)x, x\rangle^2$$

$$\leq \frac{1}{2}(\Delta - \delta) \left[\|g(A)x\|^2 - \langle g(A)x, x\rangle^2\right]^{1/2}$$

which implies that

$$\left[\|g(A)x\|^2 - \langle g(A)x, x\rangle^2\right]^{1/2} \leq \frac{1}{2}(\Delta - \delta)$$

for each $x \in H$ with $\|x\| = 1$.

This together with the first part of (3.64) proves the desired bound. ∎

The following particular cases that hold for power function are of interest:

Example 175 *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$.*

If A is positive $(m \geq 0)$ and $p, q > 0$, then

$$(0 \leq) \langle A^{p+q}x, x\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle \qquad (3.65)$$

$$\leq \frac{1}{2} \cdot (M^p - m^p) \left[\|A^q x\|^2 - \langle A^q x, x\rangle^2\right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q)\right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If A is positive definite $(m > 0)$ and $p, q < 0$, then

$$(0 \le) \left\langle A^{p+q}x, x \right\rangle - \left\langle A^p x, x \right\rangle \cdot \left\langle A^q x, x \right\rangle \tag{3.66}$$

$$\le \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} \left[\|A^q x\|^2 - \left\langle A^q x, x \right\rangle^2 \right]^{1/2}$$

$$\left[\le \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q}m^{-q}} \right]$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite $(m > 0)$ and $p < 0$, $q > 0$ then

$$(0 \le) \left\langle A^p x, x \right\rangle \cdot \left\langle A^q x, x \right\rangle - \left\langle A^{p+q}x, x \right\rangle \tag{3.67}$$

$$\le \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} \left[\|A^q x\|^2 - \left\langle A^q x, x \right\rangle^2 \right]^{1/2}$$

$$\left[\le \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} (M^q - m^q) \right]$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite $(m > 0)$ and $p > 0$, $q < 0$ then

$$(0 \le) \left\langle A^p x, x \right\rangle \cdot \left\langle A^q x, x \right\rangle - \left\langle A^{p+q}x, x \right\rangle \tag{3.68}$$

$$\le \frac{1}{2} \cdot (M^p - m^p) \left[\|A^q x\|^2 - \left\langle A^q x, x \right\rangle^2 \right]^{1/2}$$

$$\left[\le \frac{1}{4} \cdot (M^p - m^p) \frac{M^{-q} - m^{-q}}{M^{-q}m^{-q}} \right]$$

for each $x \in H$ with $\|x\| = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.65)-(3.68) follows from the Theorem 150.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 176 *Let A be a positive definite operator with $Sp(A) \subseteq [m, M]$ for some scalars $0 < m < M$.*

If $p > 0$ then

$$(0 \le) \left\langle A^p \ln Ax, x \right\rangle - \left\langle A^p x, x \right\rangle \cdot \left\langle \ln Ax, x \right\rangle \tag{3.69}$$

$$\le \begin{cases} \frac{1}{2} \cdot (M^p - m^p) \left[\|\ln Ax\|^2 - \left\langle \ln Ax, x \right\rangle^2 \right]^{1/2} \\[2mm] \ln \sqrt{\frac{M}{m}} \cdot \left[\|A^p x\|^2 - \left\langle A^p x, x \right\rangle^2 \right]^{1/2} \end{cases}$$

$$\left[\le \frac{1}{2} \cdot (M^p - m^p) \ln \sqrt{\frac{M}{m}} \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If $p < 0$ then

$$(0 \le) \langle A^p x, x \rangle \cdot \langle \ln Ax, x \rangle - \langle A^p \ln Ax, x \rangle \qquad (3.70)$$

$$\le \begin{cases} \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\|\ln Ax\|^2 - \langle \ln Ax, x \rangle^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\|A^p x\|^2 - \langle A^p x, x \rangle^2 \right]^{1/2} \end{cases}$$

$$\left[\le \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \ln \sqrt{\frac{M}{m}} \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

3.3.3 An Inequality of Grüss' Type for n Operators

The following multiple operator version of Theorem 173 holds:

Theorem 177 (Dragomir, 2008, [31]) *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous and* $\gamma := \min_{t \in [m,M]} f(t)$ *and* $\Gamma := \max_{t \in [m,M]} f(t)$ *then*

$$\left| \sum_{j=1}^{n} \langle f(A_j) g(A_j) y_j, y_j \rangle - \sum_{j=1}^{n} \langle f(A_j) y_j, y_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \quad (3.71) \right.$$

$$\left. - \frac{\gamma + \Gamma}{2} \left[\sum_{j=1}^{n} \langle g(A_j) y_j, y_j \rangle - \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right] \right|$$

$$\le \frac{1}{2} (\Gamma - \gamma) \left[\sum_{j=1}^{n} \|g(A_j) y_j\|^2 + \left(\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right)^2 \right.$$

$$\left. - 2 \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \sum_{j=1}^{n} \langle g(A_j) y_j, y_j \rangle \right]^{\frac{1}{2}}$$

for each $x_j, y_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

Proof. Follows from Theorem 173. ∎

The following particular case provides a refinement of the Mond-Pečarić result.

Corollary 178 (Dragomir, 2008, [31]) *With the assumptions of Theorem 177 we have*

$$\left| \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right| \quad (3.72)$$

$$\leq \frac{1}{2} \cdot (\Gamma - \gamma) \left[\sum_{j=1}^{n} \|g(A_j) x_j\|^2 - \left(\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right)^2 \right]^{1/2}$$

$$\left(\leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta) \right)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$ where $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$.

Example 179 *Let $A_j, j \in \{1, \ldots, n\}$ be a selfadjoint operators with $Sp(A_j) \subseteq [m, M], j \in \{1, \ldots, n\}$ for some scalars $m < M$.*
If A_j are positive $(m \geq 0)$ and $p, q > 0$, then

$$(0 \leq) \sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \quad (3.73)$$

$$\leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{j=1}^{n} \|A_j^q x_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \right)^2 \right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q) \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.
If A_j are positive definite $(m > 0)$ and $p, q < 0$, then

$$(0 \leq) \sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \quad (3.74)$$

$$\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^{n} \|A_j^q x_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \right)^2 \right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If A_j are positive definite ($m > 0$) and $p < 0$, $q > 0$ then

$$(0 \leq) \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle \qquad (3.75)$$

$$\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \right)^2 \right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left(M^q - m^q \right) \right]$$

for each $x_j \in H$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If A_j are positive definite ($m > 0$) and $p > 0$, $q < 0$ then

$$(0 \leq) \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle \qquad (3.76)$$

$$\leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \right)^2 \right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot (M^p - m^p) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x_j \in H$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.73)-(3.76) follows from the Theorem 150.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 180 *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \dots, n\}$ for some scalars $0 < m < M$.*

If $p > 0$ then

$$(0 \leq) \sum_{j=1}^{n} \langle A_j^p \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \qquad (3.77)$$

$$\leq \begin{cases} \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{j=1}^{n} \left\| \ln A_j x_j \right\|^2 - \left(\sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right)^2 \right]^{1/2} \\[2em] \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{j=1}^{n} \left\| A_j^p x_j \right\|^2 - \left(\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \right)^2 \right]^{1/2} \end{cases}$$

$$\left[\leq \frac{1}{2} \cdot (M^p - m^p) \ln \sqrt{\frac{M}{m}} \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If $p < 0$ then

$$(0 \leq) \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^p \ln A_j x_j, x_j \right\rangle \qquad (3.78)$$

$$\leq \begin{cases} \frac{1}{2} \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^{n} \|\ln A_j x_j\|^2 - \left(\sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right)^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{j=1}^{n} \|A_j^p x_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \right)^2 \right]^{1/2} \end{cases}$$

$$\left[\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \ln \sqrt{\frac{M}{m}} \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

3.3.4 Another Inequality of Grüss' Type for n Operators

The following different result for n operators can be stated as well:

Theorem 181 (Dragomir, 2008, [31]) *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$ and $\Gamma := \max_{t \in [m,M]} f(t)$ then for any $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have*

$$\left| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) y, y \right\rangle \right. \qquad (3.79)$$

$$- \frac{\gamma + \Gamma}{2} \cdot \left[\left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right]$$

$$\left. - \left\langle \sum_{k=1}^{n} p_k f(A_k) y, y \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right|$$

$$\leq \frac{\Gamma - \gamma}{2} \left[\sum_{k=1}^{n} p_k \|g(A_k) y\|^2 - 2 \left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right.$$

$$\left. + \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle^2 \right]^{1/2},$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. Follows from Theorem 177 on choosing $x_j = \sqrt{p_j} \cdot x$, $y_j = \sqrt{p_j} \cdot y$, $j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x, y \in H$, with $\|x\| = \|y\| = 1$. The details are omitted. ∎

Remark 182 *The case $n = 1$ (therefore $p = 1$) in (3.79) provides the result from Theorem 173.*

As a particular case of interest we can derive from the above theorem the following result of Grüss' type:

Corollary 183 (Dragomir, 2008, [31]) *With the assumptions of Theorem 181 we have*

$$\left| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle \right|$$

$$\leq \frac{\Gamma - \gamma}{2} \left(\sum_{k=1}^{n} p_k \|g(A_k) x\|^2 - \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle^2 \right)^{1/2} \qquad (3.80)$$

$$\left[\leq \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta) \right]$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t \in [m, M]} g(t)$ and $\Delta := \max_{t \in [m, M]} g(t)$.

Proof. It is similar with the proof from Corollary 174 and the details are omitted. ∎

The following particular cases that hold for power function are of interest:

Example 184 *Let $A_j, j \in \{1, \ldots, n\}$ be a selfadjoint operators with $Sp(A_j) \subseteq [m, M], j \in \{1, \ldots, n\}$ for some scalars $m < M$ and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$.*

If $A_j, j \in \{1, \ldots, n\}$ are positive $(m \geq 0)$ and $p, q > 0$, then

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle \qquad (3.81)$$

$$\leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{k=1}^{n} p_k \|A_k^q x\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q) \right]$$

for each $x \in H$ with $\|x\| = 1$.

If A_j, $j \in \{1, \ldots, n\}$ are positive definite $(m > 0)$ and $p, q < 0$, then

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle \qquad (3.82)$$

$$\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x \in H$ with $\|x\| = 1$.

If A_j, $j \in \{1, \ldots, n\}$ are positive definite $(m > 0)$ and $p < 0$, $q > 0$ then

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle \qquad (3.83)$$

$$\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left(M^q - m^q \right) \right]$$

for each $x \in H$ with $\|x\| = 1$.

If A_j, $j \in \{1, \ldots, n\}$ are positive definite $(m > 0)$ and $p > 0$, $q < 0$ then

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle \qquad (3.84)$$

$$\leq \frac{1}{2} \cdot \left(M^p - m^p \right) \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2}$$

$$\left[\leq \frac{1}{4} \cdot \left(M^p - m^p \right) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x \in H$ with $\|x\| = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.81)-(3.84) follows from the Theorem 150.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 185 *Let A_j, $j \in \{1, \ldots, n\}$ be positive definite operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $0 < m < M$ and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$.*

If $p > 0$ then

$$\left\langle \sum_{k=1}^n p_k A_k^p \ln A_k x, x \right\rangle - \left\langle \sum_{k=1}^n p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^n p_k \ln A_k x, x \right\rangle \quad (3.85)$$

$$\leq \begin{cases} \frac{1}{2} \cdot (M^p - m^p) \cdot \left[\sum_{k=1}^n p_k \|\ln A_k x\|^2 - \langle \sum_{k=1}^n p_k \ln A_k x, x \rangle^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{k=1}^n p_k \|A_k^p x\|^2 - \langle \sum_{k=1}^n p_k A_k^p x, x \rangle^2 \right]^{1/2} \end{cases}$$

$$\left[\leq \frac{1}{2} \cdot (M^p - m^p) \ln \sqrt{\frac{M}{m}} \right]$$

for each $x \in H$ with $\|x\| = 1$.

If $p < 0$ then

$$(0 \leq) \left\langle \sum_{k=1}^n p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^n p_k \ln A_k x, x \right\rangle \quad (3.86)$$

$$- \left\langle \sum_{k=1}^n p_k A_k^p \ln A_k x, x \right\rangle$$

$$\leq \begin{cases} \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^n p_k \|\ln A_k x\|^2 - \langle \sum_{k=1}^n p_k \ln A_k x, x \rangle^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{k=1}^n p_k \|A_k^p x\|^2 - \langle \sum_{k=1}^n p_k A_k^p x, x \rangle^2 \right]^{1/2} \end{cases}$$

$$\left[\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \ln \sqrt{\frac{M}{m}} \right]$$

for each $x \in H$ with $\|x\| = 1$.

The following norm inequalities may be stated as well:

Corollary 186 (Dragomir, 2008, [31]) *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, then for each $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n p_j = 1$ we have the norm inequality:*

$$\left\| \sum_{j=1}^n p_j f(A_j) g(A_j) \right\| \leq \left\| \sum_{j=1}^n p_j f(A_j) \right\| \cdot \left\| \sum_{j=1}^n p_j g(A_j) \right\| \quad (3.87)$$

$$+ \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta),$$

where $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$.

Proof. Utilising the inequality (3.80) we deduce the inequality

$$\left| \left\langle \sum_{k=1}^{n} p_k f\left(A_k\right) g\left(A_k\right) x, x \right\rangle \right|$$

$$\leq \left| \left\langle \sum_{k=1}^{n} p_k f\left(A_k\right) x, x \right\rangle \right| \cdot \left| \left\langle \sum_{k=1}^{n} p_k g\left(A_k\right) x, x \right\rangle \right|$$

$$+ \frac{1}{4}\left(\Gamma - \gamma\right)\left(\Delta - \delta\right)$$

for each $x \in H$ with $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ we deduce the desired inequality (3.87). ∎

Example 187 *a. Let A_j, $j \in \{1, \ldots, n\}$ be a selfadjoint operators with $Sp\left(A_j\right) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $m < M$ and $p_j \geq 0$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$.*
If A_j, $j \in \{1, \ldots, n\}$ are positive $(m \geq 0)$ and $p, q > 0$, then

$$\left\| \sum_{k=1}^{n} p_k A_k^{p+q} \right\| \leq \left\| \sum_{k=1}^{n} p_k A_k^{p} \right\| \cdot \left\| \sum_{k=1}^{n} p_k A_k^{q} \right\| \tag{3.88}$$

$$+ \frac{1}{4} \cdot \left(M^p - m^p\right)\left(M^q - m^q\right).$$

If A_j, $j \in \{1, \ldots, n\}$ are positive definite $(m > 0)$ and $p, q < 0$, then

$$\left\| \sum_{k=1}^{n} p_k A_k^{p+q} \right\| \leq \left\| \sum_{k=1}^{n} p_k A_k^{p} \right\| \cdot \left\| \sum_{k=1}^{n} p_k A_k^{q} \right\| \tag{3.89}$$

$$+ \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}}.$$

b. Let A_j, $j \in \{1, \ldots, n\}$ be positive definite operators with $Sp\left(A_j\right) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $0 < m < M$ and $p_j \geq 0$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$.
If $p > 0$ then

$$\left\| \sum_{k=1}^{n} p_k A_k^{p} \ln A_k \right\| \leq \left\| \sum_{k=1}^{n} p_k A_k^{p} \right\| \cdot \left\| \sum_{k=1}^{n} p_k \ln A_k \right\| \tag{3.90}$$

$$+ \frac{1}{2} \cdot \left(M^p - m^p\right) \ln \sqrt{\frac{M}{m}}.$$

3.4   More Inequalities of Grüss Type

3.4.1   Some Vectorial Grüss' Type Inequalities

The following lemmas, that are of interest in their own right, collect some Grüss type inequalities for vectors in inner product spaces obtained earlier by the author:

Lemma 188 (Dragomir, 2003 & 2004, [23], [28]) *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that*

$$\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \qquad \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0 \qquad (3.91)$$

or equivalently,

$$\left\| u - \frac{\alpha + \beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha|, \qquad \left\| v - \frac{\gamma + \delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma|. \qquad (3.92)$$

Then

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \qquad (3.93)$$

$$\leq \frac{1}{4} \cdot |\beta - \alpha| \, |\delta - \gamma|$$

$$- \begin{cases} [\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle]^{\frac{1}{2}}, \\[2mm] \left| \langle u, e \rangle - \frac{\alpha + \beta}{2} \right| \left| \langle v, e \rangle - \frac{\gamma + \delta}{2} \right|. \end{cases}$$

The first inequality has been obtained in [23] (see also [27, p. 44]) while the second result was established in [28] (see also [27, p. 90]). They provide refinements of the earlier result from [16] where only the first part of the bound, i.e., $\frac{1}{4} |\beta - \alpha| \, |\delta - \gamma|$ has been given. Notice that, as pointed out in [28], the upper bounds for the Grüss functional incorporated in (3.93) cannot be compared in general, meaning that one is better than the other depending on appropriate choices of the vectors and scalars involved.

Another result of this type is the following one:

Lemma 189 (Dragomir, 2004 & 2006, [24], [29]) *With the assumptions in Lemma 188 and if $\operatorname{Re} (\beta \overline{\alpha}) > 0, \operatorname{Re} (\delta \overline{\gamma}) > 0$ then*

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \qquad (3.94)$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{|\beta - \alpha| \, |\delta - \gamma|}{[\operatorname{Re}(\beta \overline{\alpha}) \operatorname{Re}(\delta \overline{\gamma})]^{\frac{1}{2}}} |\langle u, e \rangle \langle e, v \rangle|, \\[4mm] \left[\left(|\alpha + \beta| - 2 \left[\operatorname{Re} (\beta \overline{\alpha}) \right]^{\frac{1}{2}} \right) \left(|\delta + \gamma| - 2 \left[\operatorname{Re} (\delta \overline{\gamma}) \right]^{\frac{1}{2}} \right) \right]^{\frac{1}{2}} \\ \times \left[|\langle u, e \rangle \langle e, v \rangle| \right]^{\frac{1}{2}}. \end{cases}$$

The first inequality has been established in [24] (see [27, p. 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [29]. The details are omitted.

Finally, another inequality of Grüss type that has been obtained in [25] (see also [27, p. 65]) can be stated as:

Lemma 190 (Dragomir, 2004, [25]) *With the assumptions in Lemma 188 and if $\beta \neq -\alpha$, $\delta \neq -\gamma$ then*

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \tag{3.95}$$

$$\leq \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{[|\beta + \alpha| |\delta + \gamma|]^{\frac{1}{2}}} \left[(\|u\| + |\langle u, e \rangle|)(\|v\| + |\langle v, e \rangle|) \right]^{\frac{1}{2}}.$$

3.4.2 Some Inequalities of Grüss' Type for One Operator

The following results incorporates some new inequalities of Grüss' type for two functions of a selfadjoint operator.

Theorem 191 (Dragomir, 2008, [32]) *Let A be a selfadjoint operator on the Hilbert space $(H; \langle ., . \rangle)$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$ then*

$$|\langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \langle g(A) x, x \rangle| \tag{3.96}$$

$$\leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta)$$

$$- \begin{cases} [\langle \Gamma x - f(A) x, f(A) x - \gamma x \rangle \langle \Delta x - g(A) x, g(A) x - \delta x \rangle]^{\frac{1}{2}}, \\ \\ \left| \langle f(A) x, x \rangle - \frac{\Gamma + \gamma}{2} \right| \left| \langle g(A) x, x \rangle - \frac{\Delta + \delta}{2} \right|, \end{cases}$$

for each $x \in H$ with $\|x\| = 1$.

Moreover if γ and δ are positive, then we also have

$$|\langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \langle g(A) x, x \rangle| \tag{3.97}$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \langle f(A) x, x \rangle \langle g(A) x, x \rangle, \\ \\ \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right) [\langle f(A) x, x \rangle \langle g(A) x, x \rangle]^{\frac{1}{2}}, \end{cases}$$

while for $\Gamma + \gamma, \Delta + \delta \neq 0$ we have

$$|\langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \langle g(A) x, x \rangle| \tag{3.98}$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{[|\Gamma + \gamma| |\Delta + \delta|]^{\frac{1}{2}}}$$

$$\times [(\|f(A) x\| + |\langle f(A) x, x \rangle|)(\|g(A) x\| + |\langle g(A) x, x \rangle|)]^{\frac{1}{2}}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Since $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$, the by the property (P) we have that

$$\gamma \cdot 1_H \le f(A) \le \Gamma \cdot 1_H \quad \text{and} \quad \delta \cdot 1_H \le g(A) \le \Delta \cdot 1_H$$

in the operator order, which imply that

$$\begin{aligned}
[f(A) - \gamma \cdot 1][\Gamma \cdot 1_H - f(A)] &\ge 0 \text{ and} \\
[\Delta \cdot 1_H - g(A)][g(A) - \delta \cdot 1_H] &\ge 0
\end{aligned} \tag{3.99}$$

in the operator order.

We then have from (3.99)

$$\langle [f(A) - \gamma \cdot 1][\Gamma \cdot 1_H - f(A)] x, x \rangle \ge 0$$

and

$$\langle [\Delta \cdot 1_H - g(A)][g(A) - \delta \cdot 1_H] x, x \rangle \ge 0,$$

for each $x \in H$ with $\|x\| = 1$, which, by the fact that the involved operators are selfadjoint, are equivalent with the inequalities

$$\begin{aligned}
\langle \Gamma x - f(A) x, f(A) x - \gamma x \rangle &\ge 0 \text{ and} \\
\langle \Delta x - g(A) x, g(A) x - \delta x \rangle &\ge 0,
\end{aligned} \tag{3.100}$$

for each $x \in H$ with $\|x\| = 1$.

Now, if we apply Lemma 188 for $u = f(A) x$, $v = g(A) x$, $e = x$, and the real scalars Γ, γ, Δ and δ defined in the statement of the theorem, then we can state the inequality

$$|\langle f(A) x, g(A) x \rangle - \langle f(A) x, x \rangle \langle x, g(A) x \rangle| \tag{3.101}$$

$$\le \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta)$$

$$- \begin{cases} [\operatorname{Re} \langle \Gamma x - f(A) x, f(A) x - \gamma x \rangle \operatorname{Re} \langle \Delta x - g(A) x, g(A) x - \delta x \rangle]^{\frac{1}{2}}, \\ \left| \langle f(A) x, x \rangle - \frac{\Gamma + \gamma}{2} \right| \left| \langle g(A) x, x \rangle - \frac{\Delta + \delta}{2} \right|, \end{cases}$$

for each $x \in H$ with $\|x\| = 1$, which is clearly equivalent with the inequality (3.96).

The inequalities (3.97) and (3.98) follow by Lemma 189 and Lemma 190 respectively and the details are omitted. ∎

Remark 192 *The first inequality in (3.97) can be written in a more convenient way as*

$$\left| \frac{\langle f(A) g(A) x, x \rangle}{\langle f(A) x, x \rangle \langle g(A) x, x \rangle} - 1 \right| \le \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \tag{3.102}$$

for each $x \in H$ with $\|x\| = 1$, while the second inequality has the following equivalent form

$$\left| \frac{\langle f(A) g(A) x, x \rangle}{[\langle f(A) x, x \rangle \langle g(A) x, x \rangle]^{1/2}} - [\langle f(A) x, x \rangle \langle g(A) x, x \rangle]^{1/2} \right| \qquad (3.103)$$
$$\leq \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right)$$

for each $x \in H$ with $\|x\| = 1$.

We know, from [30] that if f, g are synchronous (asynchronous) functions on the interval $[m, M]$, i.e., we recall that

$$[f(t) - f(s)][g(t) - g(s)] (\geq) \leq 0 \qquad \text{for each } t, s \in [m, M],$$

then we have the inequality

$$\langle f(A) g(A) x, x \rangle \geq (\leq) \langle f(A) x, x \rangle \langle g(A) x, x \rangle \qquad (3.104)$$

for each $x \in H$ with $\|x\| = 1$, provided f, g are continuous on $[m, M]$ and A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$.

Therefore, if f, g are synchronous then we have from (3.102) and from (3.103) the following results:

$$0 \leq \frac{\langle f(A) g(A) x, x \rangle}{\langle f(A) x, x \rangle \langle g(A) x, x \rangle} - 1 \leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \qquad (3.105)$$

and

$$0 \leq \frac{\langle f(A) g(A) x, x \rangle}{[\langle f(A) x, x \rangle \langle g(A) x, x \rangle]^{1/2}} - [\langle f(A) x, x \rangle \langle g(A) x, x \rangle]^{1/2} \qquad (3.106)$$
$$\leq \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right)$$

for each $x \in H$ with $\|x\| = 1$, respectively.

If f, g are asynchronous then

$$0 \leq 1 - \frac{\langle f(A) g(A) x, x \rangle}{\langle f(A) x, x \rangle \langle g(A) x, x \rangle} \leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \qquad (3.107)$$

and

$$0 \leq [\langle f(A) x, x \rangle \langle g(A) x, x \rangle]^{1/2} - \frac{\langle f(A) g(A) x, x \rangle}{[\langle f(A) x, x \rangle \langle g(A) x, x \rangle]^{1/2}} \qquad (3.108)$$
$$\leq \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right)$$

for each $x \in H$ with $\|x\| = 1$, respectively.

It is obvious that all the inequalities from Theorem 191 can be used to obtain reverse inequalities of Grüss' type for various particular instances of operator functions, see for instance [31]. However we give here only a few provided by the inequalities (3.105) and (3.106) above.

Example 193 *Let A be a selfadjoint operator with $Sp\,(A) \subseteq [m, M]$ for some scalars $m < M$.*

If A is positive $(m \geq 0)$ and $p, q > 0$, then

$$0 \leq \frac{\langle A^{p+q}x, x \rangle}{\langle A^p x, x \rangle \cdot \langle A^q x, x \rangle} - 1 \tag{3.109}$$

$$\leq \frac{1}{4} \cdot \frac{(M^p - m^p)(M^q - m^q)}{M^{\frac{p+q}{2}} m^{\frac{p+q}{2}}}$$

and

$$0 \leq \frac{\langle A^{p+q}x, x \rangle}{[\langle A^p x, x \rangle \cdot \langle A^q x, x \rangle]^{1/2}} - [\langle A^p x, x \rangle \cdot \langle A^q x, x \rangle]^{1/2} \tag{3.110}$$

$$\leq \left(M^{\frac{p}{2}} - m^{\frac{p}{2}} \right) \left(M^{\frac{q}{2}} - m^{\frac{q}{2}} \right)$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite $(m > 0)$ and $p, q < 0$, then

$$0 \leq \frac{\langle A^{p+q}x, x \rangle}{\langle A^p x, x \rangle \cdot \langle A^q x, x \rangle} - 1 \tag{3.111}$$

$$\leq \frac{1}{4} \cdot \frac{(M^{-p} - m^{-p})(M^{-q} - m^{-q})}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}}$$

and

$$0 \leq \frac{\langle A^{p+q}x, x \rangle}{[\langle A^p x, x \rangle \cdot \langle A^q x, x \rangle]^{1/2}} - [\langle A^p x, x \rangle \cdot \langle A^q x, x \rangle]^{1/2} \tag{3.112}$$

$$\leq \frac{\left(M^{-\frac{p}{2}} - m^{-\frac{p}{2}} \right) \left(M^{-\frac{q}{2}} - m^{-\frac{q}{2}} \right)}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}}$$

for each $x \in H$ with $\|x\| = 1$.

Similar inequalities may be stated for either $p > 0, q < 0$ or $p < 0, q > 0$. The details are omitted.

Example 194 *Let A be a positive definite operator with $Sp\,(A) \subseteq [m, M]$ for some scalars $1 < m < M$. If $p > 0$ then*

$$0 \leq \frac{\langle A^p \ln Ax, x \rangle}{\langle A^p x, x \rangle \cdot \langle \ln Ax, x \rangle} - 1 \tag{3.113}$$

$$\leq \frac{1}{4} \cdot \frac{(M^p - m^p) \ln \frac{M}{m}}{M^{\frac{p}{2}} m^{\frac{p}{2}} \sqrt{\ln M \cdot \ln m}}$$

and

$$0 \leq \frac{\langle A^p \ln Ax, x \rangle}{[\langle A^p x, x \rangle \cdot \langle \ln Ax, x \rangle]^{1/2}} - [\langle A^p x, x \rangle \cdot \langle \ln Ax, x \rangle]^{1/2} \qquad (3.114)$$

$$\leq \left(M^{\frac{p}{2}} - m^{\frac{p}{2}} \right) \left[\sqrt{\ln M} - \sqrt{\ln m} \right],$$

for each $x \in H$ with $\|x\| = 1$.

3.4.3   Some Inequalities of Grüss' Type for n Operators

The following extension for sequences of operators can be stated:

Theorem 195 (Dragomir, 2008, [32]) *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$ then*

$$\left| \sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle \right|$$

$$\leq \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta) \qquad (3.115)$$

$$- \begin{cases} \left[\sum_{j=1}^n \langle \Gamma x_j - f(A_j) x_j, f(A_j) x_j - \gamma x_j \rangle \right. \\ \qquad \left. \times \sum_{j=1}^n \langle \Delta x_j - g(A_j) x_j, g(A_j) x - \delta x_j \rangle \right]^{\frac{1}{2}}, \\ \left| \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle - \frac{\Gamma + \gamma}{2} \right| \left| \sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle - \frac{\Delta + \delta}{2} \right|, \end{cases}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.
Moreover if γ and δ are positive, then we also have

$$\left| \sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle \right|$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle, \\ \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right) \qquad\qquad (3.116) \\ \times \left[\sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle \right]^{\frac{1}{2}}, \end{cases}$$

while for $\Gamma + \gamma, \Delta + \delta \neq 0$ *we have*

$$\left| \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right|$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{[|\Gamma + \gamma||\Delta + \delta|]^{\frac{1}{2}}} \tag{3.117}$$

$$\times \left[\left(\left(\sum_{j=1}^{n} \|f(A_j) x_j\|^2 \right)^{1/2} + \left| \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \right| \right) \right.$$

$$\left. \times \left(\left(\sum_{j=1}^{n} \|g(A_j) x_j\|^2 \right)^{1/2} + \left| \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right| \right) \right]^{1/2},$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. Follows from Theorem 191. The details are omitted. ∎

Remark 196 *The first inequality in (3.116) can be written in a more convenient way as*

$$\left| \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} - 1 \right| \tag{3.118}$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$, *while the second inequality has the following equivalent form*

$$\left| \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right]^{1/2}} \right. \tag{3.119}$$

$$\left. - \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right]^{1/2} \right|$$

$$\leq \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We know, from [30] that if f, g are synchronous (asynchronous) functions on the interval $[m, M]$, then we have the inequality

$$\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle \tag{3.120}$$

$$\geq (\leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, provided f, g are continuous on $[m, M]$ and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$.

Therefore, if f, g are synchronous then we have from (3.118) and from (3.119) the following results:

$$0 \leq \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} - 1 \tag{3.121}$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}}$$

and

$$0 \leq \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2}} \tag{3.122}$$

$$- \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2}$$

$$\leq \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, respectively.

If f, g are asynchronous then

$$0 \leq 1 - \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} \tag{3.123}$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}}$$

and

$$0 \le \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right]^{1/2} \tag{3.124}$$
$$- \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right]^{1/2}}$$
$$\le \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, respectively.

It is obvious that all the inequalities from Theorem 195 can be used to obtain reverse inequalities of Grüss' type for various particular instances of operator functions, see for instance [31]. However we give here only a few provided by the inequalities (3.121) and (3.122) above.

Example 197 *Let A_j $j \in \{1, \ldots, n\}$ be selfadjoint operators with $Sp(A_j) \subseteq [m, M], j \in \{1, \ldots, n\}$ for some scalars $m < M$.*
If A_j are positive ($m \ge 0$) and $p, q > 0$, then

$$0 \le \frac{\sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle}{\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle} - 1 \tag{3.125}$$
$$\le \frac{1}{4} \cdot \frac{(M^p - m^p)(M^q - m^q)}{M^{\frac{p+q}{2}} m^{\frac{p+q}{2}}}$$

and

$$0 \le \frac{\sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \right]^{1/2}} \tag{3.126}$$
$$- \left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \right]^{1/2}$$
$$\le \left(M^{\frac{p}{2}} - m^{\frac{p}{2}} \right) \left(M^{\frac{q}{2}} - m^{\frac{q}{2}} \right)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.
If A is positive definite ($m > 0$) and $p, q < 0$, then

$$0 \le \frac{\sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle}{\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle} - 1 \tag{3.127}$$
$$\le \frac{1}{4} \cdot \frac{(M^{-p} - m^{-p})(M^{-q} - m^{-q})}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}}$$

and

$$0 \leq \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^q x_j, x_j \right\rangle \right]^{1/2} \tag{3.128}$$

$$- \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x, x \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^q x_j, x_j \right\rangle \right]^{1/2}}$$

$$\leq \frac{\left(M^{-\frac{p}{2}} - m^{-\frac{p}{2}} \right) \left(M^{-\frac{q}{2}} - m^{-\frac{q}{2}} \right)}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Similar inequalities may be stated for either $p > 0, q < 0$ or $p < 0, q > 0$. The details are omitted.

Example 198 *Let A be a positive definite operator with $Sp(A) \subseteq [m, M]$ for some scalars $1 < m < M$. If $p > 0$ then*

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^p \ln A_j x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle} - 1 \tag{3.129}$$

$$\leq \frac{1}{4} \cdot \frac{(M^p - m^p) \ln \frac{M}{m}}{M^{\frac{p}{2}} m^{\frac{p}{2}} \sqrt{\ln M \cdot \ln m}}$$

and

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^p \ln A_j x_j, x_j \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle \right]^{1/2}} \tag{3.130}$$

$$- \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle \right]^{1/2}$$

$$\leq \left(M^{\frac{p}{2}} - m^{\frac{p}{2}} \right) \left[\sqrt{\ln M} - \sqrt{\ln m} \right],$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Similar inequalities may be stated for $p < 0$. The details are omitted.
The following result for n operators can be stated as well:

Corollary 199 *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$ then for any $p_j \geq 0, j \in \{1, \ldots, n\}$*

with $\sum_{j=1}^{n} p_j = 1$ we have

$$
\left| \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right|
$$

$$
\leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta) \tag{3.131}
$$

$$
- \begin{cases} \left[\sum_{j=1}^{n} p_j \langle \Gamma x - f(A_j) x, f(A_j) x - \gamma x \rangle \right. \\ \qquad \left. \times \sum_{j=1}^{n} p_j \langle \Delta x - g(A_j) x, g(A_j) x - \delta x \rangle \right]^{\frac{1}{2}}, \\ \\ \left| \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - \frac{\Gamma+\gamma}{2} \right| \left| \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle - \frac{\Delta+\delta}{2} \right|, \end{cases}
$$

for each $x \in H$, with $\|x\|^2 = 1$.
  Moreover if γ and δ are positive, then we also have

$$
\left| \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right|
$$

$$
\leq \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle, \\ \\ \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right) \\ \qquad \times \left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right]^{\frac{1}{2}}. \end{cases} \tag{3.132}
$$

while for $\Gamma + \gamma, \Delta + \delta \neq 0$ we have

$$\left| \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right|$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{[|\Gamma + \gamma| |\Delta + \delta|]^{\frac{1}{2}}} \tag{3.133}$$

$$\times \left[\left(\left(\sum_{j=1}^{n} p_j \|f(A_j) x\|^2 \right)^{1/2} + \left| \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \right| \right) \right.$$

$$\times \left. \left(\left(\sum_{j=1}^{n} p_j \|g(A_j) x\|^2 \right)^{1/2} + \left| \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right| \right) \right]^{1/2}$$

for each $x \in H$, with $\|x\|^2 = 1$.

Proof. Follows from Theorem 195 on choosing $x_j = \sqrt{p_j} \cdot x, j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ∎

Remark 200 *The first inequality in (3.132) can be written in a more convenient way as*

$$\left| \frac{\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle} - 1 \right| \tag{3.134}$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}}$$

for each $x \in H$, with $\|x\|^2 = 1$, while the second inequality has the following equivalent form

$$\left| \frac{\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle}{\left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right]^{1/2}} \right. \tag{3.135}$$

$$\left. - \left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right]^{1/2} \right|$$

$$\leq \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right)$$

for each $x \in H$, with $\|x\|^2 = 1$.

We know, from [30] that if f, g are synchronous (asynchronous) functions on the interval $[m, M]$, then we have the inequality

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \tag{3.136}$$

$$\geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle$$

for each $x \in H$, with $\|x\|^2 = 1$, provided f, g are continuous on $[m, M]$ and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M], j \in \{1, \ldots, n\}$.

Therefore, if f, g are synchronous then we have from (3.134) and from (3.135) the following results:

$$0 \leq \frac{\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle} - 1 \tag{3.137}$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}}$$

and

$$0 \leq \frac{\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle}{\left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right]^{1/2}} \tag{3.138}$$

$$- \left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right]^{1/2}$$

$$\leq \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right)$$

for each $x \in H$, with $\|x\| = 1$, respectively.

If f, g are asynchronous then

$$0 \leq 1 - \frac{\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle} \tag{3.139}$$

$$\leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}}$$

and

$$0 \leq \left[\left\langle \sum_{j=1}^{n} p_j f\left(A_j\right) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g\left(A_j\right) x, x \right\rangle \right]^{1/2} \quad (3.140)$$

$$- \frac{\left\langle \sum_{j=1}^{n} p_j f\left(A_j\right) g\left(A_j\right) x, x \right\rangle}{\left[\left\langle \sum_{j=1}^{n} p_j f\left(A_j\right) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g\left(A_j\right) x, x \right\rangle \right]^{1/2}}$$

$$\leq \left(\sqrt{\Gamma} - \sqrt{\gamma} \right) \left(\sqrt{\Delta} - \sqrt{\delta} \right)$$

for each $x \in H$, with $\|x\| = 1$, respectively.

The above inequalities (3.137) - (3.140) can be used to state various particular inequalities as in the previous examples, however the details are left to the interested reader.

3.5   More Inequalities for the Čebyšev Functional

3.5.1   A Refinement and Some Related Results

The following result can be stated:

Theorem 201 (Dragomir, 2008, [33]) *Let A be a selfadjoint operator with $Sp\left(A\right) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous with $\delta := \min_{t \in [m,M]} g\left(t\right)$ and $\Delta := \max_{t \in [m,M]} g\left(t\right)$, then*

$$|C\left(f, g; A; x\right)| \quad (3.141)$$

$$\leq \frac{1}{2} \left(\Delta - \delta\right) \left\langle \left| f\left(A\right) - \left\langle f\left(A\right) x, x \right\rangle \cdot 1_H \right| x, x \right\rangle$$

$$\leq \frac{1}{2} \left(\Delta - \delta\right) C^{1/2}\left(f, f; A; x\right),$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since $\delta := \min_{t \in [m,M]} g\left(t\right)$ and $\Delta := \max_{t \in [m,M]} g\left(t\right)$, we have

$$\left| g\left(t\right) - \frac{\Delta + \delta}{2} \right| \leq \frac{1}{2} \left(\Delta - \delta\right), \quad (3.142)$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

If we multiply the inequality (3.142) with $|f\left(t\right) - \left\langle f\left(A\right) x, x \right\rangle|$ we get

$$\left| f\left(t\right) g\left(t\right) - \left\langle f\left(A\right) x, x \right\rangle g\left(t\right) - \frac{\Delta + \delta}{2} f\left(t\right) + \frac{\Delta + \delta}{2} \left\langle f\left(A\right) x, x \right\rangle \right| \quad (3.143)$$

$$\leq \frac{1}{2} \left(\Delta - \delta\right) |f\left(t\right) - \left\langle f\left(A\right) x, x \right\rangle|,$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

Now, if we apply the property (P) for the inequality (3.143) and a selfad-joint operator B with $Sp(B) \subset [m, M]$, then we get the following inequality of interest in itself:

$$\left| \langle f(B) g(B) y, y \rangle - \langle f(A) x, x \rangle \langle g(B) y, y \rangle \right. \tag{3.144}$$
$$\left. - \frac{\Delta + \delta}{2} \langle f(B) y, y \rangle + \frac{\Delta + \delta}{2} \langle f(A) x, x \rangle \right|$$
$$\leq \frac{1}{2} (\Delta - \delta) \langle |f(B) - \langle f(A) x, x \rangle \cdot 1_H | y, y \rangle,$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

If we choose in (3.144) $y = x$ and $B = A$, then we deduce the first inequality in (3.141).

Now, by the Schwarz inequality in H we have

$$\langle |f(A) - \langle f(A) x, x \rangle \cdot 1_H | x, x \rangle$$
$$\leq \| |f(A) - \langle f(A) x, x \rangle \cdot 1_H | x \|$$
$$= \| f(A) x - \langle f(A) x, x \rangle \cdot x \|$$
$$= \left[\| f(A) x \|^2 - \langle f(A) x, x \rangle^2 \right]^{1/2}$$
$$= C^{1/2}(f, f; A; x),$$

for any $x \in H$ with $\|x\| = 1$, and the second part of (3.141) is also proved. ∎

Let U be a selfadjoint operator on the Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following representation in terms of the Riemann-Stieltjes integral:

$$\langle f(U) x, x \rangle = \int_{m-0}^{M} f(\lambda) d(\langle E_\lambda x, x \rangle), \tag{3.145}$$

for any $x \in H$ with $\|x\| = 1$. The function $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* on the interval $[m, M]$ and

$$g_x(m - 0) = 0 \quad \text{and} \quad g_x(M) = 1 \tag{3.146}$$

for any $x \in H$ with $\|x\| = 1$.

The following result is of interest:

Theorem 202 (Dragomir, 2008, [33]) *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type, i.e., for a given $r \in (0, 1]$ and $L > 0$ we have*

$$|f(s) - f(t)| \leq L |s - t|^r \quad \text{for any } s, t \in [m, M],$$

then we have the Ostrowski type inequality for selfadjoint operators:

$$|f(s) - \langle f(A)x, x \rangle| \leq L \left[\frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right| \right]^r, \qquad (3.147)$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.
   Moreover, we have

$$|\langle f(B)y, y \rangle - \langle f(A)x, x \rangle| \qquad (3.148)$$
$$\leq \langle |f(B) - \langle f(A)x, x \rangle \cdot 1_H| y, y \rangle$$
$$\leq L \left[\frac{1}{2}(M-m) + \left\langle \left| B - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle \right]^r,$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. We use the following Ostrowski type inequality for the Riemann-Stieltjes integral obtained by the author in [22]:

$$\left| f(s)[u(b) - u(a)] - \int_a^b f(t) \, du(t) \right| \qquad (3.149)$$

$$\leq L \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right]^r \bigvee_a^b (u)$$

for any $s \in [a, b]$, provided that f is of $r - L-$Hölder type on $[a, b]$, u is of bounded variation on $[a, b]$ and $\bigvee_a^b (u)$ denotes the total variation of u on $[a, b]$.
   Now, applying this inequality for $u(\lambda) = g_x(\lambda) := \langle E_\lambda x, x \rangle$ where $x \in H$ with $\|x\| = 1$ we get

$$\left| f(s) - \int_{m-0}^M f(\lambda) \, d(\langle E_\lambda x, x \rangle) \right| \qquad (3.150)$$

$$\leq L \left[\frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right| \right]^r \bigvee_{m-0}^M (g_x)$$

which, by (3.145) and (3.146) is equivalent with (3.147).
   By applying the property (P) for the inequality (3.147) and the operator B we have

$$\langle |f(B) - \langle f(A)x, x \rangle \cdot 1_H| y, y \rangle$$
$$\leq L \left\langle \left[\frac{1}{2}(M-m) + \left| B - \frac{m+M}{2} \cdot 1_H \right| \right]^r y, y \right\rangle$$
$$\leq L \left\langle \left[\frac{1}{2}(M-m) + \left| B - \frac{m+M}{2} \right| \cdot 1_H \right] y, y \right\rangle^r$$
$$= L \left[\frac{1}{2}(M-m) + \left\langle \left| B - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle \right]^r$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which proves the second inequality in (3.148).

Further, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [44, p. 5]) applied for the modulus, we can state that

$$|\langle h(A) x, x\rangle| \le \langle |h(A)| x, x\rangle \qquad \text{(M)}$$

for any $x \in H$ with $\|x\| = 1$, where h is a continuous function on $[m, M]$.

Now, if we apply the inequality (M), then we have

$$|\langle [f(B) - \langle f(A) x, x\rangle \cdot 1_H] y, y\rangle| \le \langle |f(B) - \langle f(A) x, x\rangle \cdot 1_H| y, y\rangle$$

which shows the first part of (3.148), and the proof is complete. ∎

Remark 203 *With the above assumptions for f, A and B we have the following particular inequalities of interest:*

$$\left| f\left(\frac{m+M}{2}\right) - \langle f(A) x, x\rangle \right| \le \frac{1}{2^r} L (M-m)^r \qquad (3.151)$$

and

$$|f(\langle Ax, x\rangle) - \langle f(A) x, x\rangle| \qquad (3.152)$$
$$\le L \left[\frac{1}{2}(M-m) + \left| \langle Ax, x\rangle - \frac{m+M}{2}\right| \right]^r,$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequalities:

$$|\langle f(A) y, y\rangle - \langle f(A) x, x\rangle| \qquad (3.153)$$
$$\le \langle |f(A) - \langle f(A) x, x\rangle \cdot 1_H| y, y\rangle$$
$$\le L \left[\frac{1}{2}(M-m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| y, y\right\rangle \right]^r,$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$,

$$|\langle [f(B) - f(A)] x, x\rangle| \qquad (3.154)$$
$$\le \langle |f(B) - \langle f(A) x, x\rangle \cdot 1_H| x, x\rangle$$
$$\le L \left[\frac{1}{2}(M-m) + \left\langle \left| B - \frac{m+M}{2} \cdot 1_H \right| x, x\right\rangle \right]^r$$

and, more particularly,

$$\langle |f(A) - \langle f(A) x, x\rangle \cdot 1_H| x, x\rangle \qquad (3.155)$$
$$\le L \left[\frac{1}{2}(M-m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x\right\rangle \right]^r,$$

for any $x \in H$ with $\|x\| = 1$.

We also have the norm inequality

$$\|f(B) - f(A)\| \leq L \left[\frac{1}{2}(M - m) + \left\| B - \frac{m + M}{2} \cdot 1_H \right\| \right]^r. \qquad (3.156)$$

The following corollary of the above Theorem 202 can be useful for applications:

Corollary 204 (Dragomir, 2008, [33]) *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is absolutely continuous then we have the Ostrowski type inequality for selfadjoint operators:*

$$|f(s) - \langle f(A)x, x \rangle| \qquad (3.157)$$

$$\leq \begin{cases} \left[\frac{1}{2}(M - m) + \left| s - \frac{m+M}{2} \right| \right] \|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty[m, M]; \\[2ex] \left[\frac{1}{2}(M - m) + \left| s - \frac{m+M}{2} \right| \right]^{1/q} \|f'\|_{p,[m,M]} & \text{if } f' \in L_p[m, M], \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$, where $\|\cdot\|_{p,[m,M]}$ are the Lebesgue norms, i.e.,

$$\|h\|_{\infty,[m,M]} := ess \sup_{t \in [m,M]} \|h(t)\|$$

and

$$\|h\|_{p,[m,M]} := \left(\int_m^M |h(t)|^p \right)^{1/p}, \quad p \geq 1.$$

Moreover, we have

$$|\langle f(B)y, y \rangle - \langle f(A)x, x \rangle| \qquad (3.158)$$
$$\leq \langle |f(B) - \langle f(A)x, x \rangle \cdot 1_H| y, y \rangle$$
$$\leq \begin{cases} \left[\frac{M-m}{2} + \left\langle \left| B - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle \right] \|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty[m, M]; \\[2ex] \left[\frac{M-m}{2} + \left\langle \left| B - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle \right]^{\frac{1}{q}} \|f'\|_{p,[m,M]} & \text{if } f' \in L_p[m, M], \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, on utilising Theorem 201 we can provide the following upper bound for the Čebyšev functional that may be more useful in applications:

Corollary 205 (Dragomir, 2008, [33]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $g : [m, M] \longrightarrow \mathbb{R}$*

is continuous with $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$, then for any $f : [m, M] \longrightarrow \mathbb{R}$ of $r - L-$ Hölder type we have the inequality:

$$|C(f, g; A; x)| \tag{3.159}$$

$$\leq \frac{1}{2} (\Delta - \delta) L \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m + M}{2} \cdot 1_H \right| x, x \right\rangle \right]^r,$$

for any $x \in H$ with $\|x\| = 1$.

Remark 206 *With the assumptions from Corollary 205 for g and A and if f is absolutely continuos on $[m, M]$, then we have the inequalities:*

$$|C(f, g; A; x)| \tag{3.160}$$

$$\leq \frac{1}{2} (\Delta - \delta)$$

$$\times \begin{cases} \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle \right] \|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty [m, M]; \\\\ \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle \right]^{1/q} \|f'\|_{p,[m,M]} & \text{if } f' \in L_\infty [m, M], \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

3.5.2   Some Inequalities for Sequences of Operators

Consider the sequence of selfadjoint operators $\mathbf{A} = (A_1, \ldots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ are such that $\sum_{j=1}^n \|x_j\|^2 = 1$, then we can consider the following Čebyšev type functional

$$C(f, g; \mathbf{A}, \mathbf{x}) := \sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle$$

$$- \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle.$$

As a particular case of the above functional and for a probability sequence $\mathbf{p} = (p_1, \ldots, p_n)$, i.e., $p_j \geq 0$ for $j \in \{1, \ldots, n\}$ and $\sum_{j=1}^n p_j = 1$, we can also consider the functional

$$C(f, g; \mathbf{A}, \mathbf{p}, x) := \left\langle \sum_{j=1}^n p_j f(A_j) g(A_j) x, x \right\rangle$$

$$- \left\langle \sum_{j=1}^n p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle$$

where $x \in H$, $\|x\| = 1$.

We know, from [30] that for the sequence of selfadjoint operators $\mathbf{A} = (A_1, \ldots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for the synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the inequality

$$C(f, g; \mathbf{A}, \mathbf{x}) \geq (\leq) 0 \tag{3.161}$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Also, for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ and any $x \in H$, $\|x\| = 1$ we have

$$C(f, g; \mathbf{A}, \mathbf{p}, x) \geq (\leq) 0. \tag{3.162}$$

On the other hand, the following Grüss' type inequality is valid as well [31]:

$$|C(f, g; \mathbf{A}, \mathbf{x})| \leq \frac{1}{2} \cdot (\Gamma - \gamma) [C(g, g; \mathbf{A}, \mathbf{x})]^{1/2} \tag{3.163}$$

$$\left(\leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta) \right)$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, where f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m, M]} f(t)$, $\Gamma := \max_{t \in [m, M]} f(t)$, $\delta := \min_{t \in [m, M]} g(t)$ and $\Delta := \max_{t \in [m, M]} g(t)$.

Similarly, for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ and any $x \in H$, $\|x\| = 1$ we also have the inequality:

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \leq \frac{1}{2} \cdot (\Gamma - \gamma) [C(g, g; \mathbf{A}, \mathbf{p}, x)]^{1/2} \tag{3.164}$$

$$\left(\leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta) \right).$$

We can state now the following new result:

Theorem 207 (Dragomir, 2008, [33]) *Consider the sequence of selfadjoint operators* $\mathbf{A} = (A_1, \ldots, A_n)$ *with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous with* $\delta := \min_{t \in [m, M]} g(t)$ *and* $\Delta := \max_{t \in [m, M]} g(t)$, *then*

$$|C(f, g; \mathbf{A}; \mathbf{x})| \tag{3.165}$$

$$\leq \frac{1}{2} (\Delta - \delta) \sum_{j=1}^{n} \left\langle \left| f(A_j) - \sum_{k=1}^{n} \langle f(A_k) x_k, x_k \rangle \cdot 1_H \right| x_j, x_j \right\rangle$$

$$\leq \frac{1}{2} (\Delta - \delta) C^{1/2}(f, f; \mathbf{A}; \mathbf{x}),$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ *such that* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. Follows from Theorem 201 and the details are omitted. ∎

The following particular results is of interest for applications:

Corollary 208 (Dragomir, 2008, [33]) *Consider the sequence of self-adjoint operators* $\mathbf{A} = (A_1, \ldots, A_n)$ *with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous with* $\delta := \min_{t \in [m,M]} g(t)$ *and* $\Delta := \max_{t \in [m,M]} g(t)$, *then for any* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^n p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$ *we have*

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \tag{3.166}$$

$$\leq \frac{1}{2}(\Delta - \delta)\left\langle \sum_{j=1}^n p_j \left| f(A_j) - \left\langle \sum_{k=1}^n p_k f(A_k) x, x \right\rangle \cdot 1_H \right| x, x \right\rangle$$

$$\leq \frac{1}{2}(\Delta - \delta) C^{1/2}(f, f; \mathbf{A}, \mathbf{p}, x).$$

Proof. In we choose in Theorem 207 $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^n p_j = 1$ and $x \in H$, with $\|x\| = 1$ then a simple calculation shows that the inequality (3.165) becomes (3.166). The details are omitted. ∎

In a similar manner we can prove the following result as well:

Theorem 209 (Dragomir, 2008, [33]) *Consider the sequences of self-adjoint operators* $\mathbf{A} = (A_1, \ldots, A_n)$, $\mathbf{B} = (B_1, \ldots, B_n)$ *with* $Sp(A_j), Sp(B_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is of* $r - L-$*Hölder type, then we have the Ostrowski type inequality for sequences of selfadjoint operators:*

$$\left| f(s) - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \right| \tag{3.167}$$

$$\leq L\left[\frac{1}{2}(M - m) + \left| s - \frac{m + M}{2} \right|\right]^r,$$

for any $s \in [m, M]$ *and any* $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ *such that* $\sum_{j=1}^n \|x_j\|^2 = 1$.

Moreover, we have

$$\left| \sum_{j=1}^n \langle f(B_j) y_j, y_j \rangle - \sum_{k=1}^n \langle f(A_k) x_k, x_k \rangle \right| \tag{3.168}$$

$$\leq \sum_{j=1}^n \left\langle \left| f(B_j) - \sum_{k=1}^n \langle f(A_k) x_k, x_k \rangle \cdot 1_H \right| y_j, y_j \right\rangle$$

$$\leq L\left[\frac{1}{2}(M - m) + \sum_{j=1}^n \left\langle \left| B_j - \frac{m + M}{2} \cdot 1_H \right| y_j, y_j \right\rangle\right]^r,$$

for any $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in H^n$ such that $\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n \|y_j\|^2 = 1$.

Corollary 210 (Dragomir, 2008, [33]) *Consider the sequences of self-adjoint operators* $\mathbf{A} = (A_1, \ldots, A_n), \mathbf{B} = (B_1, \ldots, B_n)$ *with* $Sp(A_j), Sp(B_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is of* $r - L -$*Hölder type, then for any* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^n p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$ *we have the weighted Ostrowski type inequality for sequences of selfadjoint operators:*

$$\left| f(s) - \left\langle \sum_{j=1}^n p_j f(A_j) x, x \right\rangle \right| \tag{3.169}$$

$$\leq L \left[\frac{1}{2} (M - m) + \left| s - \frac{m + M}{2} \right| \right]^r,$$

for any $s \in [m, M]$.
   Moreover, we have

$$\left| \left\langle \sum_{j=1}^n q_j f(B_j) y, y \right\rangle - \left\langle \sum_{k=1}^n p_k f(A_k) x, x \right\rangle \right| \tag{3.170}$$

$$\leq \left\langle \sum_{j=1}^n q_j \left| f(B_j) - \left\langle \sum_{k=1}^n p_k f(A_k) x, x \right\rangle \cdot 1_H \right| y, y \right\rangle$$

$$\leq L \left[\frac{1}{2} (M - m) + \left\langle \sum_{j=1}^n q_j \left| B_j - \frac{m + M}{2} \cdot 1_H \right| y, y \right\rangle \right]^r,$$

for any $q_k \geq 0, k \in \{1, \ldots, n\}$ *with* $\sum_{k=1}^n q_k = 1$ *and* $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

3.5.3   Some Reverses of Jensen's Inequality

It is clear that all the above inequalities can be applied for various particular instances of functions f and g. However, in the following we only consider the inequalities

$$|f(\langle Ax, x \rangle) - \langle f(A) x, x \rangle| \tag{3.171}$$

$$\leq L \left[\frac{1}{2} (M - m) + \left| \langle Ax, x \rangle - \frac{m + M}{2} \right| \right]^r$$

for any $x \in H$ with $\|x\| = 1$, where the function $f : [m, M] \to \mathbb{R}$ is of $r - L-$Hölder type, and

$$|f(\langle Ax, x \rangle)) - \langle f(A)x, x \rangle| \tag{3.172}$$

$$\leq \begin{cases} \left[\frac{1}{2}(M-m) + \left|\langle Ax, x \rangle - \frac{m+M}{2}\right|\right] \|f'\|_{\infty,[m,M]}, & \text{if } f' \in L_\infty[m, M] \\[3mm] \left[\frac{1}{2}(M-m) + \left|\langle Ax, x \rangle - \frac{m+M}{2}\right|\right]^q \|f'\|_{p,[m,M]}, & \text{if } f' \in L_p[m, M]; \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$$

for any $x \in H$ with $\|x\| = 1$, where the function $f : [m, M] \to \mathbb{R}$ is absolutely continuous on $[m, M]$, which are related to the *Jensen's inequality* for convex functions.

1. Now, if we consider the concave function $f : [m, M] \subset [0, \infty) \to \mathbb{R}$, $f(t) = t^r$ with $r \in (0, 1)$ and take into account that it is of $r - L-$Hölder type with the constant $L = 1$, then from (3.171) we derive the following reverse for the *Hölder-McCarthy inequality [48]*

$$0 \leq \langle A^r x, x \rangle - \langle Ax, x \rangle^r \leq \left[\frac{1}{2}(M-m) + \left|\langle Ax, x \rangle - \frac{m+M}{2}\right|\right]^r \tag{3.173}$$

for any $x \in H$ with $\|x\| = 1$.

2. Now, if we consider the functions $f : [m, M] \subset (0, \infty) \to \mathbb{R}$ with $f(t) = t^s$ and $s \in (-\infty, 0) \cup (0, \infty)$, then they are absolutely continuous and

$$\|f'\|_{\infty,[m,M]} = \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\[3mm] |s| m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1). \end{cases}$$

If $p \geq 1$, then

$$\|f'\|_{p,[m,M]} = |s| \left(\int_m^M t^{p(s-1)} dt\right)^{1/p}$$

$$= |s| \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p} \\[3mm] \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}. \end{cases}$$

On making use of the first inequality from (3.172) we deduce for a given $s \in (-\infty, 0) \cup (0, \infty)$ that

$$|\langle Ax, x \rangle^s - \langle A^s x, x \rangle| \tag{3.174}$$

$$\leq \left[\frac{1}{2}(M-m) + \left|\langle Ax, x \rangle - \frac{m+M}{2}\right|\right]$$

$$\times \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\[3mm] |s| m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1). \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

The second part of (3.172) will produce the following reverse of the *Hölder-McCarthy inequality* as well:

$$|\langle Ax, x \rangle^s - \langle A^s x, x \rangle| \tag{3.175}$$

$$\leq |s| \left[\frac{1}{2} (M - m) + \left| \langle Ax, x \rangle - \frac{m + M}{2} \right| \right]^q$$

$$\times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1} \right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p} \\ \\ \left[\ln \left(\frac{M}{m} \right) \right]^{1/p} & \text{if } s = 1 - \frac{1}{p} \end{cases}$$

for any $x \in H$ with $\|x\| = 1$, where $s \in (-\infty, 0) \cup (0, \infty)$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

3. Now, if we consider the function $f(t) = \ln t$ defined on the interval $[m, M] \subset (0, \infty)$, then f is also absolutely continuous and

$$\|f'\|_{p,[m,M]} = \begin{cases} m^{-1} & \text{for } p = \infty, \\ \\ \left(\frac{M^{p-1} - m^{p-1}}{(p-1)M^{p-1}m^{p-1}} \right)^{1/p} & \text{for } p > 1, \\ \\ \ln \left(\frac{M}{m} \right) & \text{for } p = 1. \end{cases}$$

Making use of the first inequality in (3.172) we deduce

$$0 \leq \ln \left(\langle Ax, x \rangle \right) - \langle \ln (A) x, x \rangle \tag{3.176}$$

$$\leq \left[\frac{1}{2} (M - m) + \left| \langle Ax, x \rangle - \frac{m + M}{2} \right| \right] m^{-1}$$

and

$$0 \leq \ln \left(\langle Ax, x \rangle \right) - \langle \ln (A) x, x \rangle \tag{3.177}$$

$$\leq \left[\frac{1}{2} (M - m) + \left| \langle Ax, x \rangle - \frac{m + M}{2} \right| \right]^q \left(\frac{M^{p-1} - m^{p-1}}{(p - 1) M^{p-1} m^{p-1}} \right)^{1/p}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Similar results can be stated for sequences of operators, however the details are left to the interested reader.

3.5.4 Some Particular Grüss' Type Inequalities

In this last section we provide some particular cases that can be obtained via the Grüss' type inequalities established before. For this purpose we select only two examples as follows.

Let A be a selfadjoint operator with $Sp\,(A) \subseteq [m, M]$ for some real numbers $m < M$. If $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g\,(t)$ and $\Delta := \max_{t\in[m,M]} g\,(t)$, then for any $f : [m, M] \longrightarrow \mathbb{R}$ of $r - L-$Hölder type we have the inequality:

$$|\langle f\,(A)\,g\,(A)\,x, x\rangle - \langle f\,(A)\,x, x\rangle \cdot \langle g\,(A)\,x, x\rangle| \qquad (3.178)$$
$$\leq \frac{1}{2}\,(\Delta - \delta)\,L\left[\frac{1}{2}\,(M - m) + \left\langle \left|A - \frac{m + M}{2} \cdot 1_H\right| x, x\right\rangle\right]^r,$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if f is absolutely continuos on $[m, M]$, then we have the inequalities:

$$|\langle f\,(A)\,g\,(A)\,x, x\rangle - \langle f\,(A)\,x, x\rangle \cdot \langle g\,(A)\,x, x\rangle| \qquad (3.179)$$
$$\leq \frac{1}{2}\,(\Delta - \delta)$$
$$\times \begin{cases} \left[\frac{1}{2}\,(M - m) + \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| x, x\right\rangle\right]\|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty\,[m, M]\,; \\[2mm] \left[\frac{1}{2}\,(M - m) + \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| x, x\right\rangle\right]^{1/q}\|f'\|_{p,[m,M]} & \text{if } f' \in L_p\,[m, M]\,, \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

1. If we consider the concave function $f : [m, M] \subset [0, \infty) \to \mathbb{R}$, $f\,(t) = t^r$ with $r \in (0, 1)$ and take into account that it is of $r - L-$Hölder type with the constant $L = 1$, then from (3.178) we derive the following result:

$$|\langle A^r g\,(A)\,x, x\rangle - \langle A^r x, x\rangle \cdot \langle g\,(A)\,x, x\rangle| \qquad (3.180)$$
$$\leq \frac{1}{2}\,(\Delta - \delta)\left[\frac{1}{2}\,(M - m) + \left\langle \left|A - \frac{m + M}{2} \cdot 1_H\right| x, x\right\rangle\right]^r,$$

for any $x \in H$ with $\|x\| = 1$, where $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g\,(t)$ and $\Delta := \max_{t\in[m,M]} g\,(t)$.

Now, consider the function $g : [m, M] \subset (0, \infty) \to \mathbb{R}$, $g\,(t) = t^p$ with $p \in (-\infty, 0) \cup (0, \infty)$. Obviously,

$$\Delta - \delta = \begin{cases} M^p - m^p & \text{if } p > 0, \\[2mm] \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} & \text{if } p < 0, \end{cases}$$

and by (3.180) we get for any $x \in H$ with $\|x\| = 1$ that

$$0 \leq \left\langle A^{r+p} x, x\right\rangle - \langle A^r x, x\rangle \cdot \langle A^p x, x\rangle \qquad (3.181)$$
$$\leq \frac{1}{2}\,(M^p - m^p)\left[\frac{1}{2}\,(M - m) + \left\langle \left|A - \frac{m + M}{2} \cdot 1_H\right| x, x\right\rangle\right]^r,$$

when $p > 0$ and

$$0 \le \langle A^r x, x \rangle \cdot \langle A^p x, x \rangle - \langle A^{r+p} x, x \rangle \tag{3.182}$$
$$\le \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m + M}{2} \cdot 1_H \right| x, x \right\rangle \right]^r,$$

when $p < 0$.

If $g : [m, M] \subset (0, \infty) \to \mathbb{R}$, $g(t) = \ln t$, then by (3.180) we also get the inequality for logarithm:

$$0 \le \langle A^r \ln Ax, x \rangle - \langle A^r x, x \rangle \cdot \langle \ln Ax, x \rangle \tag{3.183}$$
$$\le \ln \sqrt{\frac{M}{m}} \cdot \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m + M}{2} \cdot 1_H \right| x, x \right\rangle \right]^r,$$

for any $x \in H$ with $\|x\| = 1$.

2. Now consider the functions $f, g : [m, M] \subset (0, \infty) \to \mathbb{R}$, with $f(t) = t^s$ and $g(t) = t^w$ with $s, w \in (-\infty, 0) \cup (0, \infty)$. We have

$$\|f'\|_{\infty, [m, M]} = \begin{cases} s M^{s-1} & \text{for } s \in [1, \infty), \\ |s| \, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1). \end{cases}$$

and, for $p \ge 1$,

$$\|f'\|_{p, [m, M]} = |s| \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1} \right)^{1/p} & \text{if } s \ne 1 - \frac{1}{p} \\ \left[\ln \left(\frac{M}{m} \right) \right]^{1/p} & \text{if } s = 1 - \frac{1}{p}. \end{cases}$$

If $w > 0$, then by the first inequality in (3.179) we have

$$\left| \langle A^{s+w} x, x \rangle - \langle A^s x, x \rangle \cdot \langle A^w x, x \rangle \right| \tag{3.184}$$
$$\le \frac{1}{2} (M^w - m^w) \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m + M}{2} \cdot 1_H \right| x, x \right\rangle \right]$$
$$\times \begin{cases} s M^{s-1} & \text{for } s \in [1, \infty), \\ |s| \, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1). \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

If $w < 0$, then by the same inequality we also have

$$\left| \langle A^{s+w} x, x \rangle - \langle A^s x, x \rangle \cdot \langle A^w x, x \rangle \right| \tag{3.185}$$
$$\le \frac{1}{2} \cdot \frac{M^{-w} - m^{-w}}{M^{-w} m^{-w}} \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m + M}{2} \cdot 1_H \right| x, x \right\rangle \right]$$
$$\times \begin{cases} s M^{s-1} & \text{for } s \in [1, \infty), \\ |s| \, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1), \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Finally, if we assume that $p > 1$ and $w > 0$, then by the second inequality in (3.179) we have

$$\left| \langle A^{s+w} x, x \rangle - \langle A^s x, x \rangle \cdot \langle A^w x, x \rangle \right| \tag{3.186}$$

$$\leq \frac{1}{2} |s| \left(M^w - m^w \right) \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle \right]^{1/q}$$

$$\times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1} \right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p} \\ \\ \left[\ln \left(\frac{M}{m} \right) \right]^{1/p} & \text{if } s = 1 - \frac{1}{p}, \end{cases}$$

while for $w < 0$, we also have

$$\left| \langle A^{s+w} x, x \rangle - \langle A^s x, x \rangle \cdot \langle A^w x, x \rangle \right| \tag{3.187}$$

$$\leq \frac{1}{2} |s| \cdot \frac{M^{-w} - m^{-w}}{M^{-w} m^{-w}} \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle \right]^{1/q}$$

$$\times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1} \right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p} \\ \\ \left[\ln \left(\frac{M}{m} \right) \right]^{1/p} & \text{if } s = 1 - \frac{1}{p}, \end{cases}$$

where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in H$ with $\|x\| = 1$.

3.6   Bounds for the Čebyšev Functional of Lipschitzian Functions

3.6.1   The Case of Lipschitzian Functions

The following result can be stated:

Theorem 211 (Dragomir, 2008, [34]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is Lipschitzian with the constant $L > 0$ and $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$, then*

$$|C(f, g; A; x)| \tag{3.188}$$

$$\leq \frac{1}{2} (\Delta - \delta) L \langle \ell_{A,x}(A) x, x \rangle \leq (\Delta - \delta) L C(e, e; A; x)$$

for any $x \in H$ with $\|x\| = 1$, where

$$\ell_{A,x}(t) := \langle |t \cdot 1_H - A| x, x \rangle$$

is a continuous function on $[m, M]$, $e(t) = t$ *and*

$$C(e, e; A; x) = \|Ax\|^2 - \langle Ax, x \rangle^2 \; (\geq 0).$$ (3.189)

Proof. First of all, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [44, p. 5]) applied for the modulus, we can state that

$$|\langle h(A) x, x \rangle| \leq \langle |h(A)| x, x \rangle$$ (M)

for any $x \in H$ with $\|x\| = 1$, where h is a continuous function on $[m, M]$.

Since f is Lipschitzian with the constant $L > 0$, then for any $t, s \in [m, M]$ we have

$$|f(t) - f(s)| \leq L |t - s|.$$ (3.190)

Now, if we fix $t \in [m, M]$ and apply the property (P) for the inequality (3.190) and the operator A we get

$$\langle |f(t) \cdot 1_H - f(A)| x, x \rangle \leq L \langle |t \cdot 1_H - A| x, x \rangle,$$ (3.191)

for any $x \in H$ with $\|x\| = 1$.

Utilising the property (M) we get

$$|f(t) - \langle f(A) x, x \rangle| = |\langle f(t) \cdot 1_H - f(A) x, x \rangle| \leq \langle |f(t) \cdot 1_H - f(A)| x, x \rangle$$

which together with (3.191) gives

$$|f(t) - \langle f(A) x, x \rangle| \leq L \ell_{A,x}(t)$$ (3.192)

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

Since $\delta := \min_{t \in [m, M]} g(t)$ and $\Delta := \max_{t \in [m, M]} g(t)$, we also have

$$\left| g(t) - \frac{\Delta + \delta}{2} \right| \leq \frac{1}{2} (\Delta - \delta)$$ (3.193)

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

If we multiply the inequality (3.192) with (3.193) we get

$$\left| f(t) g(t) - \langle f(A) x, x \rangle g(t) - \frac{\Delta + \delta}{2} f(t) + \frac{\Delta + \delta}{2} \langle f(A) x, x \rangle \right|$$ (3.194)

$$\leq \frac{1}{2} (\Delta - \delta) L \ell_{A,x}(t) = \frac{1}{2} (\Delta - \delta) L \langle |t \cdot 1_H - A| x, x \rangle$$

$$\leq \frac{1}{2} (\Delta - \delta) L \left\langle |t \cdot 1_H - A|^2 x, x \right\rangle^{1/2}$$

$$= \frac{1}{2} (\Delta - \delta) L \left(\langle A^2 x, x \rangle - 2 \langle Ax, x \rangle t + t^2 \right)^{1/2},$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

Now, if we apply the property (P) for the inequality (3.194) and a selfadjoint operator B with $Sp(B) \subset [m, M]$, then we get the following inequality of interest in itself:

$$\langle f(B) g(B) y, y \rangle - \langle f(A) x, x \rangle \langle g(B) y, y \rangle \qquad (3.195)$$
$$-\frac{\Delta + \delta}{2} \langle f(B) y, y \rangle + \frac{\Delta + \delta}{2} \langle f(A) x, x \rangle \Big|$$
$$\leq \frac{1}{2} (\Delta - \delta) L \langle \ell_{A,x} (B) y, y \rangle$$
$$\leq \frac{1}{2} (\Delta - \delta) L \left\langle \left(\langle A^2 x, x \rangle 1_H - 2 \langle Ax, x \rangle B + B^2 \right)^{1/2} y, y \right\rangle$$
$$\leq \frac{1}{2} (\Delta - \delta) L \left(\langle A^2 x, x \rangle - 2 \langle Ax, x \rangle \langle By, y \rangle + \langle B^2 y, y \rangle \right),$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, if we choose in (3.195) $y = x$ and $B = A$, then we deduce the desired result (3.188). ∎

In the case of two Lipschitzian functions, the following result may be stated as well:

Theorem 212 (Dragomir, 2008, [34]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are Lipschitzian with the constants $L, K > 0$, then*

$$|C(f, g; A; x)| \leq LKC(e, e; A; x), \qquad (3.196)$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since $f, g : [m, M] \longrightarrow \mathbb{R}$ are Lipschitzian, then

$$|f(t) - f(s)| \leq L|t - s| \text{ and } |g(t) - g(s)| \leq K|t - s|$$

for any $t, s \in [m, M]$, which gives the inequality

$$|f(t) g(t) - f(t) g(s) - f(s) g(t) + f(s) g(s)| \leq KL \left(t^2 - 2ts + s^2 \right)$$

for any $t, s \in [m, M]$.

Now, fix $t \in [m, M]$ and if we apply the properties (P) and (M) for the operator A we get successively

$$|f(t) g(t) - \langle g(A) x, x \rangle f(t) \qquad (3.197)$$
$$- \langle f(A) x, x \rangle g(t) + \langle f(A) g(A) x, x \rangle|$$
$$= |\langle [f(t) g(t) \cdot 1_H - f(t) g(A) - f(A) g(t) + f(A) g(A)] x, x \rangle|$$
$$\leq \langle |f(t) g(t) \cdot 1_H - f(t) g(A) - f(A) g(t) + f(A) g(A)| x, x \rangle$$
$$\leq KL \left\langle \left(t^2 \cdot 1_H - 2tA + A^2 \right) x, x \right\rangle$$
$$= KL \left(t^2 - 2t \langle Ax, x \rangle + \langle A^2 x, x \rangle \right)$$

for any $x \in H$ with $\|x\| = 1$.

Further, fix $x \in H$ with $\|x\| = 1$. On applying the same properties for the inequality (3.197) and another selfadjoint operator B with $Sp(B) \subset [m, M]$, we have

$$
\begin{aligned}
&|\langle f(B) g(B) y, y \rangle - \langle g(A) x, x \rangle \langle f(B) y, y \rangle \\
&- \langle f(A) x, x \rangle \langle g(B) y, y \rangle + \langle f(A) g(A) x, x \rangle|
\end{aligned}
\tag{3.198}
$$

$$
\begin{aligned}
&= |\langle [f(B) g(B) - \langle g(A) x, x \rangle f(B) \\
&\quad - \langle f(A) x, x \rangle g(B) + \langle f(A) g(A) x, x \rangle 1_H] y, y \rangle| \\
&\leq \langle |f(B) g(B) - \langle g(A) x, x \rangle f(B) \\
&\quad - \langle f(A) x, x \rangle g(B) + \langle f(A) g(A) x, x \rangle 1_H| y, y \rangle \\
&\leq KL \left\langle \left(B^2 - 2 \langle Ax, x \rangle B + \langle A^2 x, x \rangle 1_H \right) y, y \right\rangle \\
&= KL \left(\langle B^2 y, y \rangle - 2 \langle Ax, x \rangle \langle By, y \rangle + \langle A^2 x, x \rangle \right)
\end{aligned}
$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which is an inequality of interest in its own right.

Finally, on making $B = A$ and $y = x$ in (3.198) we deduce the desired result (3.196). ∎

3.6.2 Some Inequalities for Sequences of Operators

Consider the sequence of selfadjoint operators $\mathbf{A} = (A_1, \ldots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ are such that $\sum_{j=1}^n \|x_j\|^2 = 1$, then we can consider the following Čebyšev type functional

$$
C(f, g; \mathbf{A}, \mathbf{x}) := \sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j \rangle
$$

$$
- \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j \rangle.
$$

As a particular case of the above functional and for a probability sequence $\mathbf{p} = (p_1, \ldots, p_n)$, i.e., $p_j \geq 0$ for $j \in \{1, \ldots, n\}$ and $\sum_{j=1}^n p_j = 1$, we can also consider the functional

$$
C(f, g; \mathbf{A}, \mathbf{p}, x) := \left\langle \sum_{j=1}^n p_j f(A_j) g(A_j) x, x \right\rangle
$$

$$
- \left\langle \sum_{j=1}^n p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle
$$

where $x \in H$, $\|x\| = 1$.

We know, from [30] that for the sequence of selfadjoint operators $\mathbf{A} = (A_1, \ldots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for the synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the inequality

$$C(f, g; \mathbf{A}, \mathbf{x}) \geq (\leq) 0 \tag{3.199}$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Also, for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ and any $x \in H$, $\|x\| = 1$ we have

$$C(f, g; \mathbf{A}, \mathbf{p}, x) \geq (\leq) 0. \tag{3.200}$$

On the other hand, the following Grüss' type inequality is valid as well [30]:

$$|C(f, g; \mathbf{A}, \mathbf{x})| \leq \frac{1}{2} \cdot (\Gamma - \gamma) [C(g, g; \mathbf{A}, \mathbf{x})]^{1/2} \tag{3.201}$$

$$\left(\leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta) \right)$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, where f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m, M]} f(t)$, $\Gamma := \max_{t \in [m, M]} f(t)$, $\delta := \min_{t \in [m, M]} g(t)$ and $\Delta := \max_{t \in [m, M]} g(t)$.

Similarly, for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ and any $x \in H$, $\|x\| = 1$ we also have the inequality:

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \leq \frac{1}{2} \cdot (\Gamma - \gamma) [C(g, g; \mathbf{A}, \mathbf{p}, x)]^{1/2} \tag{3.202}$$

$$\left(\leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta) \right).$$

We can state now the following new result:

Theorem 213 (Dragomir, 2008, [34]) *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is Lipschitzian with the constant* $L > 0$ *and* $g : [m, M] \longrightarrow \mathbb{R}$ *is continuous with* $\delta := \min_{t \in [m, M]} g(t)$ *and* $\Delta := \max_{t \in [m, M]} g(t)$, *then*

$$|C(f, g; \mathbf{A}, \mathbf{x})| \leq \frac{1}{2} (\Delta - \delta) L \sum_{k=1}^{n} \langle \ell_{\mathbf{A}, \mathbf{x}}(A_k) x_k, x_k \rangle \tag{3.203}$$

$$\leq (\Delta - \delta) LC(e, e; \mathbf{A}; \mathbf{x})$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$, *where*

$$\ell_{\mathbf{A}, \mathbf{x}}(t) := \sum_{j=1}^{n} \langle |t \cdot 1_H - A_j| \, x_j, x_j \rangle$$

is a continuous function on $[m, M]$, $e(t) = t$ *and*

$$C(e, e; \mathbf{A}; \mathbf{x}) = \sum_{j=1}^{n} \|Ax_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^2 \ (\geq 0).$$

Proof. Follows from Theorem 211. The details are omitted. ∎

As a particular case we have:

Corollary 214 (Dragomir, 2008, [34]) *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is Lipschitzian with the constant* $L > 0$ *and* $g : [m, M] \longrightarrow \mathbb{R}$ *is continuous with* $\delta := \min_{t \in [m, M]} g(t)$ *and* $\Delta := \max_{t \in [m, M]} g(t)$, *then for any* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$ *we have*

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \leq \frac{1}{2} (\Delta - \delta) L \left\langle \sum_{k=1}^{n} p_k \ell_{\mathbf{A}, \mathbf{p}, x}(A_k) x, x \right\rangle \qquad (3.204)$$

$$\leq (\Delta - \delta) L C(e, e; \mathbf{A}, \mathbf{p}, x)$$

where

$$\ell_{\mathbf{A}, \mathbf{p}, x}(t) := \left\langle \sum_{j=1}^{n} p_j |t \cdot 1_H - A_j| x, x \right\rangle$$

is a continuous function on $[m, M]$ *and*

$$C(e, e; \mathbf{A}, \mathbf{p}, x) = \sum_{j=1}^{n} p_j \|Ax_j\|^2 - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^2 \ (\geq 0).$$

Proof. In we choose in Theorem 213 $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$ then a simple calculation shows that the inequality (3.203) becomes (3.204). The details are omitted. ∎

In a similar way we obtain the following results as well:

Theorem 215 (Dragomir, 2008, [34]) *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are Lipschitzian with the constants* $L, K > 0$, *then*

$$|C(f, g; \mathbf{A}, \mathbf{x})| \leq LKC(e, e; \mathbf{A}, \mathbf{x}), \qquad (3.205)$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Corollary 216 *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *be a sequence of selfadjoint opera-*
tors with $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$.
If $f, g : [m, M] \longrightarrow \mathbb{R}$ *are Lipschitzian with the constants* $L, K > 0$, *then*
for any $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *we have*

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \leq LKC(e, e; \mathbf{A}, \mathbf{p}, x), \qquad (3.206)$$

for any $x \in H$ *with* $\|x\| = 1$.

3.6.3 The Case of $(\varphi, \Phi) - Lipschitzian$ Functions

The following lemma may be stated.

Lemma 217 *Let* $u : [a, b] \to \mathbb{R}$ *and* $\varphi, \Phi \in \mathbb{R}$ *with* $\Phi > \varphi$. *The following*
statements are equivalent:

(i) *The function* $u - \frac{\varphi + \Phi}{2} \cdot e$, *where* $e(t) = t, t \in [a, b]$, *is* $\frac{1}{2}(\Phi - \varphi) - Lipschitzian$;

(ii) *We have the inequality:*

$$\varphi \leq \frac{u(t) - u(s)}{t - s} \leq \Phi \quad \text{for each} \quad t, s \in [a, b] \quad \text{with } t \neq s; \quad (3.207)$$

(iii) *We have the inequality:*

$$\varphi(t - s) \leq u(t) - u(s) \leq \Phi(t - s) \quad \text{for each } t, s \in [a, b] \text{ with } t > s.$$
$$(3.208)$$

Following [47], we can introduce the concept:

Definition 218 *The function* $u : [a, b] \to \mathbb{R}$ *which satisfies one of the*
equivalent conditions (i) – (iii) is said to be $(\varphi, \Phi) - Lipschitzian$ *on* $[a, b]$.

Notice that in [47], the definition was introduced on utilising the state-
ment (iii) and only the equivalence (i) $\Leftrightarrow$ (iii) was considered.
Utilising *Lagrange's mean value theorem*, we can state the following result
that provides practical examples of $(\varphi, \Phi) - Lipschitzian$ functions.

Proposition 219 *Let* $u : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differen-*
tiable on (a, b). *If*

$$-\infty < \gamma := \inf_{t \in (a,b)} u'(t), \qquad \sup_{t \in (a,b)} u'(t) =: \Gamma < \infty \qquad (3.209)$$

then u *is* $(\gamma, \Gamma) - Lipschitzian$ *on* $[a, b]$.

The following result can be stated:

Theorem 220 (Dragomir, 2008, [34]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is $(\varphi, \Phi)-$Lipschitzian on $[a, b]$ and $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$, then*

$$\left| C(f, g; A; x) - \frac{\varphi + \Phi}{2} C(e, g; A; x) \right| \tag{3.210}$$

$$\leq \frac{1}{4}(\Delta - \delta)(\Phi - \varphi) \langle \ell_{A,x}(A) x, x \rangle$$

$$\leq \frac{1}{2}(\Delta - \delta)(\Phi - \varphi) C(e, e; A; x)$$

for any $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 211 applied for the $\frac{1}{2}(\Phi - \varphi)-$Lipschitzian function $f - \frac{\varphi + \Phi}{2} \cdot e$ (see Lemma 217) and the details are omitted.

Theorem 221 (Dragomir, 2008, [34]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $f, g : [m, M] \longrightarrow \mathbb{R}$. If f is $(\varphi, \Phi)-$Lipschitzian and g is $(\psi, \Psi)-$Lipschitzian on $[a, b]$, then*

$$\left| C(f, g; A; x) - \frac{\Phi + \varphi}{2} C(e, g; A; x) \right. \tag{3.211}$$

$$\left. - \frac{\Psi + \psi}{2} C(f, e; A; x) + \frac{\Phi + \varphi}{2} \cdot \frac{\Psi + \psi}{2} C(e, e; A; x) \right|$$

$$\leq \frac{1}{4}(\Phi - \varphi)(\Psi - \psi) C(e, e; A; x),$$

for any $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 212 applied for the $\frac{1}{2}(\Phi - \varphi)-$Lipschitzian function $f - \frac{\varphi + \Phi}{2} \cdot e$ and the $\frac{1}{2}(\Psi - \psi)-$Lipschitzian function $g - \frac{\Psi + \psi}{2} \cdot e$. The details are omitted.

Similar results can be derived for sequences of operators, however they will not be presented here.

3.6.4   Some Applications

It is clear that all the inequalities obtained in the previous sections can be applied to obtain particular inequalities of interest for different selections of the functions f and g involved. However we will present here only some particular results that can be derived from the inequality

$$|C(f, g; A; x)| \leq LKC(e, e; A; x), \tag{3.212}$$

that holds for the Lipschitzian functions f and g, the first with the constant $L > 0$ and the second with the constant $K > 0$.

1. Now, if we consider the functions $f, g : [m, M] \subset (0, \infty) \to \mathbb{R}$ with $f(t) = t^p, g(t) = t^q$ and $p, q \in (-\infty, 0) \cup (0, \infty)$ then they are Lipschitzian with the constants $L = \|f'\|_\infty$ and $K = \|g'\|_\infty$. Since $f'(t) = pt^{p-1}, g(t) = qt^{q-1}$, hence

$$\|f'\|_\infty = \begin{cases} pM^{p-1} & \text{for } p \in [1, \infty), \\ |p|\, m^{p-1} & \text{for } p \in (-\infty, 0) \cup (0, 1) \end{cases}$$

and

$$\|g'\|_\infty = \begin{cases} qM^{q-1} & \text{for } q \in [1, \infty), \\ |q|\, m^{q-1} & \text{for } q \in (-\infty, 0) \cup (0, 1) \end{cases}.$$

Therefore we can state the following inequalities for the powers of a positive definite operator A with $Sp(A) \subset [m, M] \subset (0, \infty)$.

If $p, q \geq 1$, then

$$\begin{aligned}
(0 \quad \leq \quad) & \langle A^{p+q}x, x \rangle - \langle A^p x, x \rangle \cdot \langle A^q x, x \rangle \\
\leq \quad & pqM^{p+q-2} \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)
\end{aligned} \tag{3.213}$$

for each $x \in H$ with $\|x\| = 1$.

If $p \geq 1$ and $q \in (-\infty, 0) \cup (0, 1)$, then

$$\begin{aligned}
& \left| \langle A^{p+q}x, x \rangle - \langle A^p x, x \rangle \cdot \langle A^q x, x \rangle \right| \\
\leq \quad & p\,|q|\, M^{p-1} m^{q-1} \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)
\end{aligned} \tag{3.214}$$

for each $x \in H$ with $\|x\| = 1$.

If $p \in (-\infty, 0) \cup (0, 1)$ and $q \geq 1$, then

$$\begin{aligned}
& \left| \langle A^{p+q}x, x \rangle - \langle A^p x, x \rangle \cdot \langle A^q x, x \rangle \right| \\
\leq \quad & |p|\, qM^{q-1} m^{p-1} \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)
\end{aligned} \tag{3.215}$$

for each $x \in H$ with $\|x\| = 1$.

If $p, q \in (-\infty, 0) \cup (0, 1)$, then

$$\begin{aligned}
& \left| \langle A^{p+q}x, x \rangle - \langle A^p x, x \rangle \cdot \langle A^q x, x \rangle \right| \\
\leq \quad & |pq|\, m^{p+q-2} \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)
\end{aligned} \tag{3.216}$$

for each $x \in H$ with $\|x\| = 1$.

Moreover, if we take $p = 1$ and $q = -1$ in (3.214), then we get the following result

$$(0 \leq) \langle Ax, x \rangle \cdot \langle A^{-1}x, x \rangle - 1 \leq m^{-2} \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right) \tag{3.217}$$

for each $x \in H$ with $\|x\| = 1$.

2. Consider now the functions $f, g : [m, M] \subset (0, \infty) \to \mathbb{R}$ with $f(t) = t^p, p \in (-\infty, 0) \cup (0, \infty)$ and $g(t) = \ln t$. Then g is also Lipschitzian with the constant $K = \|g'\|_\infty = m^{-1}$. Applying the inequality (3.212) we then have for any $x \in H$ with $\|x\| = 1$ that

$$(0 \leq) \langle A^p \ln Ax, x \rangle - \langle A^p x, x \rangle \cdot \langle \ln Ax, x \rangle \tag{3.218}$$
$$\leq pM^{p-1} m^{-1} \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)$$

if $p \geq 1$,

$$(0 \leq) \langle A^p \ln Ax, x \rangle - \langle A^p x, x \rangle \cdot \langle \ln Ax, x \rangle \tag{3.219}$$
$$\leq pm^{p-2} \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)$$

if $p \in (0, 1)$ and

$$(0 \leq) \langle A^p x, x \rangle \cdot \langle \ln Ax, x \rangle - \langle A^p \ln Ax, x \rangle \tag{3.220}$$
$$\leq (-p) m^{p-2} \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)$$

if $p \in (-\infty, 0)$.

3. Now consider the functions $f, g : [m, M] \subset \mathbb{R} \to \mathbb{R}$ given by $f(t) = \exp(\alpha t)$ and $g(t) = \exp(\beta t)$ with α, β nonzero real numbers. It is obvious that

$$\|f'\|_\infty = |\alpha| \times \begin{cases} \exp(\alpha M) & \text{for } \alpha > 0, \\ \exp(\alpha m) & \text{for } \alpha < 0 \end{cases}$$

and

$$\|g'\|_\infty = |\beta| \times \begin{cases} \exp(\beta M) & \text{for } \beta > 0, \\ \exp(\beta m) & \text{for } \beta < 0 \end{cases}.$$

Finally, on applying the inequality (3.212) we get

$$(0 \leq) \langle \exp[(\alpha + \beta) A] x, x \rangle - \langle \exp(\alpha A) x, x \rangle \cdot \langle \exp(\beta A) x, x \rangle$$
$$\leq |\alpha\beta| \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right) \times \begin{cases} \exp[(\alpha + \beta) M] & \text{for } \alpha, \beta > 0, \\ \exp[(\alpha + \beta) m] & \text{for } \alpha, \beta < 0 \end{cases}$$

and

$$(0 \leq) \langle \exp(\alpha A) x, x \rangle \cdot \langle \exp(\beta A) x, x \rangle - \langle \exp[(\alpha + \beta) A] x, x \rangle$$
$$\leq |\alpha\beta| \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right) \times \begin{cases} \exp(\alpha M + \beta m) & \text{for } \alpha > 0, \beta < 0 \\ \exp(\alpha m + \beta M) & \text{for } \alpha < 0, \beta > 0 \end{cases}$$

for each $x \in H$ with $\|x\| = 1$.

3.7 Quasi Grüss' Type Inequalities

3.7.1 Introduction

In [16], in order to generalize the above result in abstract structures the author has proved the following Grüss' type inequality in real or complex inner product spaces.

Theorem 222 (Dragomir, 1999, [16]) *Let $(H, \langle ., . \rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e \in H, \|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \qquad (3.221)$$

hold, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|. \qquad (3.222)$$

The constant $\frac{1}{4}$ is best possible in the sense that it can not be replaced by a smaller constant.

For other results of this type, see the recent monograph [27] and the references therein.

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation theorem* in terms of the Riemann-Stieltjes integral:

$$f(U) = \int_{m-0}^{M} f(\lambda) \, dE_\lambda, \qquad (3.223)$$

which in terms of vectors can be written as

$$\langle f(U) x, y \rangle = \int_{m-0}^{M} f(\lambda) \, d \langle E_\lambda x, y \rangle, \qquad (3.224)$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m - 0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

3.7.2   Vector Inequalities

In this section we provide various bounds for the magnitude of the difference

$$\langle f(A) x, y \rangle - \langle x, y \rangle \langle f(A) x, x \rangle$$

under different assumptions on the continuous function, the selfadjoint operator $A : H \to H$ and the vectors $x, y \in H$ with $\|x\| = 1$.

Theorem 223 (Dragomir, 2010, [35]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Assume that $x, y \in H, \|x\| = 1$ are such that there exists $\gamma, \Gamma \in \mathbb{C}$ with either*

$$\operatorname{Re} \langle \Gamma x - y, y - \gamma x \rangle \geq 0 \tag{3.225}$$

or, equivalently

$$\left\| y - \frac{\gamma + \Gamma}{2} x \right\| \leq \frac{1}{2} |\Gamma - \gamma| .$$

1. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality

$$|\langle f(A) x, y \rangle - \langle x, y \rangle \langle f(A) x, x \rangle| \tag{3.226}$$

$$\leq \max_{\lambda \in [m, M]} |\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle| \bigvee_m^M (f)$$

$$\leq \max_{\lambda \in [m, M]} (\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle)^{1/2} \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \bigvee_m^M (f)$$

$$\leq \frac{1}{2} \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \bigvee_m^M (f) \leq \frac{1}{4} |\Gamma - \gamma| \bigvee_m^M (f) .$$

2. If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on $[m, M]$, then we have the inequality

$$|\langle f(A) x, y \rangle - \langle x, y \rangle \langle f(A) x, x \rangle| \tag{3.227}$$

$$\leq L \int_{m-0}^M |\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle| \, d\lambda$$

$$\leq L \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \int_{m-0}^M (\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle)^{1/2} \, d\lambda$$

$$\leq L \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \langle (M 1_H - A) x, x \rangle^{1/2} \langle (A - m 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{2} (M - m) L \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \leq \frac{1}{4} |\Gamma - \gamma| (M - m) L .$$

3. If $f : [m, M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on $[m, M]$, then we have the inequality

$$|\langle f(A) x, y \rangle - \langle x, y \rangle \langle f(A) x, x \rangle| \tag{3.228}$$

$$\leq \int_{m-0}^{M} |\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle| \, df(\lambda)$$

$$\leq \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \int_{m-0}^{M} \left(\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \right)^{1/2} df(\lambda)$$

$$\leq \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2}$$

$$\times \langle (f(M) 1_H - f(A)) x, x \rangle^{1/2} \langle (f(A) - f(m) 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{2} [f(M) - f(m)] \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \leq \frac{1}{4} |\Gamma - \gamma| [f(M) - f(m)].$$

Proof. First of all, we notice that by the Schwarz inequality in H we have for any $u, v, e \in H$ with $\|e\| = 1$ that

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \left(\|u\|^2 - |\langle u, e \rangle|^2 \right)^{1/2} \left(\|v\|^2 - |\langle v, e \rangle|^2 \right)^{1/2}. \tag{3.229}$$

Now on utilizing (3.229), we can state that

$$|\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle| \tag{3.230}$$

$$\leq \left(\|E_\lambda x\|^2 - |\langle E_\lambda x, x \rangle|^2 \right)^{1/2} \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2}$$

for any $\lambda \in [m, M]$.

Since E_λ are projections and $E_\lambda \geq 0$ then

$$\|E_\lambda x\|^2 - |\langle E_\lambda x, x \rangle|^2 = \langle E_\lambda x, x \rangle - \langle E_\lambda x, x \rangle^2 \tag{3.231}$$

$$= \langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \leq \frac{1}{4}$$

for any $\lambda \in [m, M]$ and $x \in H$ with $\|x\| = 1$.

Also, by making use of the Grüss' type inequality in inner product spaces obtained by the author in [16] we have

$$\left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \leq \frac{1}{2} |\Gamma - \gamma|. \tag{3.232}$$

Combining the relations (3.230)-(3.232) we deduce the following inequality that is of interest in itself

$$|\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle| \tag{3.233}$$

$$\leq \left(\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \right)^{1/2} \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2}$$

$$\leq \frac{1}{2} \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \leq \frac{1}{4} |\Gamma - \gamma|$$

for any $\lambda \in [m, M]$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v), \tag{3.234}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising this property of the Riemann-Stieltjes integral and the inequality (3.233) we have

$$\left| \int_{m-0}^M [\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle] \, df(\lambda) \right| \tag{3.235}$$

$$\leq \max_{\lambda \in [m,M]} |\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle| \bigvee_m^M (f)$$

$$\leq \max_{\lambda \in [m,M]} \left(\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \right)^{1/2} \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \bigvee_m^M (f)$$

$$\leq \frac{1}{2} \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \bigvee_m^M (f) \leq \frac{1}{4} |\Gamma - \gamma| \bigvee_m^M (f)$$

for x and y as in the assumptions of the theorem.

Now, integrating by parts in the Riemann-Stieltjes integral and making use of the spectral representation theorem we have

$$\int_{m-0}^M [\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle] \, df(\lambda) \tag{3.236}$$

$$= [\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle] f(\lambda) \big|_{m-0}^M$$

$$- \int_{m-0}^M f(\lambda) \, d[\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle]$$

$$= \langle x, y \rangle \int_{m-0}^M f(\lambda) \, d \langle E_\lambda x, x \rangle - \int_{m-0}^M f(\lambda) \, d \langle E_\lambda x, y \rangle$$

$$= \langle x, y \rangle \langle f(A) x, x \rangle - \langle f(A) x, y \rangle$$

which together with (3.235) produces the desired result (3.226).

Now, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\,dv(t) \right| \le L \int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have from (3.233) that

$$\left| \int_{m-0}^M [\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle\,\langle x, y\rangle]\,df(\lambda) \right| \tag{3.237}$$

$$\le L \int_{m-0}^M |\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle\,\langle x, y\rangle|\,d\lambda$$

$$\le L \left(\|y\|^2 - |\langle y, x\rangle|^2 \right)^{1/2} \int_{m-0}^M (\langle E_\lambda x, x\rangle\,\langle (1_H - E_\lambda) x, x\rangle)^{1/2}\,d\lambda.$$

If we use the Cauchy-Bunyakovsky-Schwarz integral inequality and the spectral representation theorem we have successively

$$\int_{m-0}^M (\langle E_\lambda x, x\rangle\,\langle (1_H - E_\lambda) x, x\rangle)^{1/2}\,d\lambda \tag{3.238}$$

$$\le \left[\int_{m-0}^M \langle E_\lambda x, x\rangle\,d\lambda \right]^{1/2} \left[\int_{m-0}^M \langle (1_H - E_\lambda) x, x\rangle\,d\lambda \right]^{1/2}$$

$$= \left[\langle E_\lambda x, x\rangle\,\lambda|_{m-0}^M - \int_{m-0}^M \lambda d\,\langle E_\lambda x, x\rangle \right]^{1/2}$$

$$\times \left[\langle (1_H - E_\lambda) x, x\rangle\,\lambda|_{m-0}^M - \int_{m-0}^M \lambda d\,\langle (1_H - E_\lambda) x, x\rangle \right]$$

$$= \langle (M1_H - A) x, x\rangle^{1/2}\,\langle (A - m1_H) x, x\rangle^{1/2}.$$

On utilizing (3.238), (3.237) and (3.236) we deduce the first three inequalities in (3.227).

The fourth inequality follows from the fact that

$$\langle (M1_H - A) x, x\rangle\,\langle (A - m1_H) x, x\rangle$$

$$\le \frac{1}{4} [\langle (M1_H - A) x, x\rangle + \langle (A - m1_H) x, x\rangle]^2 = \frac{1}{4}(M - m)^2.$$

The last part follows from (3.232).

Further, from the theory of Riemann-Stieltjes integral it is also well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes in-

tegrals $\int_a^b p(t) \, dv(t)$ and $\int_a^b |p(t)| \, dv(t)$ exist and

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \int_a^b |p(t)| \, dv(t). \tag{3.239}$$

Utilising this property and the inequality (3.233) we have successively

$$\left| \int_{m-0}^M \left[\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle \right] df(\lambda) \right| \tag{3.240}$$

$$\leq \int_{m-0}^M \left| \langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle \right| df(\lambda)$$

$$\leq \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \int_{m-0}^M \left(\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \right)^{1/2} df(\lambda).$$

Applying the Cauchy-Bunyakovsky-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic integrators and the spectral representation theorem we have

$$\int_{m-0}^M \left(\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \right)^{1/2} df(\lambda) \tag{3.241}$$

$$\leq \left[\int_{m-0}^M \langle E_\lambda x, x \rangle \, df(\lambda) \right]^{1/2} \left[\int_{m-0}^M \langle (1_H - E_\lambda) x, x \rangle \, df(\lambda) \right]^{1/2}$$

$$= \left[\langle E_\lambda x, x \rangle f(\lambda) \big|_{m-0}^M - \int_{m-0}^M f(\lambda) \, d \langle E_\lambda x, x \rangle \right]^{1/2}$$

$$\times \left[\langle (1_H - E_\lambda) x, x \rangle f(\lambda) \big|_{m-0}^M - \int_{m-0}^M f(\lambda) \, d \langle (1_H - E_\lambda) x, x \rangle \right]^{1/2}$$

$$= \langle (f(M) 1_H - f(A)) x, x \rangle^{1/2} \langle (f(A) - f(m) 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{2} [f(M) - f(m)]$$

and the proof is complete. ∎

Remark 224 *If we drop the conditions on x, y, we can obtain from the inequalities (3.226)-(3.227) the following results that can be easily applied for particular functions:*

1. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality

$$\left| \langle f(A) x, y \rangle \|x\|^2 - \langle x, y \rangle \langle f(A) x, x \rangle \right| \tag{3.242}$$

$$\leq \frac{1}{2} \|x\|^2 \left(\|y\|^2 \|x\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \bigvee_m^M (f)$$

for any $x, y \in H, x \neq 0$.

2. *If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$\left| \langle f(A) x, y \rangle \|x\|^2 - \langle x, y \rangle \langle f(A) x, x \rangle \right| \tag{3.243}$$

$$\leq L \left(\|y\|^2 \|x\|^2 - |\langle y, x \rangle|^2 \right)^{1/2}$$

$$\times \left[\langle (M 1_H - A) x, x \rangle \langle (A - m 1_H) x, x \rangle \right]^{1/2}$$

$$\leq \frac{1}{2} (M - m) L \|x\|^2 \left(\|y\|^2 \|x\|^2 - |\langle y, x \rangle|^2 \right)^{1/2}$$

for any $x, y \in H, x \neq 0$.

3. *If $f : [m, M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on $[m, M]$, then we have the inequality*

$$\left| \langle f(A) x, y \rangle \|x\|^2 - \langle x, y \rangle \langle f(A) x, x \rangle \right| \tag{3.244}$$

$$\leq \left(\|y\|^2 \|x\|^2 - |\langle y, x \rangle|^2 \right)^{1/2}$$

$$\times \left[\langle (f(M) 1_H - f(A)) x, x \rangle \langle (f(A) - f(m) 1_H) x, x \rangle \right]^{1/2}$$

$$\leq \frac{1}{2} [f(M) - f(m)] \|x\|^2 \left(\|y\|^2 \|x\|^2 - |\langle y, x \rangle|^2 \right)^{1/2}$$

for any $x, y \in H, x \neq 0$.

We are able now to provide the following corollary:

Corollary 225 (Dragomir, 2010, [35]) *With the assumptions of Theorem 223 and if $f : [m, M] \to \mathbb{R}$ is a (φ, Φ)-Lipschitzian function then we have*

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \langle f(A) x, x \rangle \right| \tag{3.245}$$

$$\leq \frac{1}{2} (\Phi - \varphi) \int_{m-0}^{M} |\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{2} (\Phi - \varphi) \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2} \int_{m-0}^{M} \left(\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \right)^{1/2} d\lambda$$

$$\leq \frac{1}{2} (\Phi - \varphi) \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2}$$

$$\times \langle (M 1_H - A) x, x \rangle^{1/2} \langle (A - m 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{4} (M - m) (\Phi - \varphi) \left(\|y\|^2 - |\langle y, x \rangle|^2 \right)^{1/2}$$

$$\leq \frac{1}{8} |\Gamma - \gamma| (M - m) (\Phi - \varphi).$$

The proof follows from the second part of Theorem 223 applied for the $\frac{1}{2}(\Phi - \varphi)$-Lipschitzian function $f - \frac{\Phi + \varphi}{2} \cdot e$ by performing the required calculations in the first term of the inequality. The details are omitted.

3.7.3   Applications for Grüss' Type Inequalities

The following result provides some Grüss' type inequalities for two function of two selfadjoint operators.

Proposition 226 (Dragomir, 2010, [35]) *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. Assume that $g : [m, M] \to \mathbb{R}$ is a continuous function and denote $n := \min_{t \in [m, M]} g(t)$ and $N := \max_{t \in [m, M]} g(t)$.*

1. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality

$$|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle g(B)x, x\rangle| \tag{3.246}$$

$$\leq \max_{\lambda \in [m, M]} |\langle E_\lambda x, g(B)x\rangle - \langle E_\lambda x, x\rangle \langle x, g(B)x\rangle| \bigvee_m^M (f)$$

$$\leq \max_{\lambda \in [m, M]} (\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle)^{1/2}$$

$$\times \left(\|g(B)x\|^2 - |\langle g(B)x, x\rangle|^2\right)^{1/2} \bigvee_m^M (f)$$

$$\leq \frac{1}{2}\left(\|g(B)x\|^2 - |\langle g(B)x, x\rangle|^2\right)^{1/2} \bigvee_m^M (f) \leq \frac{1}{4}(N - n)\bigvee_m^M (f)$$

for any $x \in H, \|x\| = 1$.

2. *If* $f : [m, M] \to \mathbb{C}$ *is a Lipschitzian function with the constant* $L > 0$ *on* $[m, M]$, *then we have the inequality*

$$|\langle f(A) x, g(B) x \rangle - \langle f(A) x, x \rangle \langle g(B) x, x \rangle| \qquad (3.247)$$

$$\leq L \int_{m-0}^{M} |\langle E_\lambda x, g(B) x \rangle - \langle E_\lambda x, x \rangle \langle x, g(B) x \rangle| \, d\lambda$$

$$\leq L \left(\|g(B) x\|^2 - |\langle g(B) x, x \rangle|^2 \right)^{1/2}$$

$$\times \int_{m-0}^{M} (\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle)^{1/2} \, d\lambda$$

$$\leq L \left(\|g(B) x\|^2 - |\langle g(B) x, x \rangle|^2 \right)^{1/2}$$

$$\times \langle (M 1_H - A) x, x \rangle^{1/2} \langle (A - m 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{2} (M - m) L \left(\|g(B) x\|^2 - |\langle g(B) x, x \rangle|^2 \right)^{1/2}$$

$$\leq \frac{1}{4} (N - n)(M - m) L$$

for any $x \in H, \|x\| = 1.$

3. *If* $f : [m, M] \to \mathbb{R}$ *is a continuous monotonic nondecreasing function on* $[m, M]$, *then we have the inequality*

$$|\langle f(A) x, g(B) x \rangle - \langle f(A) x, x \rangle \langle g(B) x, x \rangle| \qquad (3.248)$$

$$\leq \int_{m-0}^{M} |\langle E_\lambda x, g(B) x \rangle - \langle E_\lambda x, x \rangle \langle x, g(B) x \rangle| \, df(\lambda)$$

$$\leq \left(\|g(B) x\|^2 - |\langle g(B) x, x \rangle|^2 \right)^{1/2}$$

$$\times \int_{m-0}^{M} (\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle)^{1/2} \, df(\lambda)$$

$$\leq \left(\|g(B) x\|^2 - |\langle g(B) x, x \rangle|^2 \right)^{1/2}$$

$$\times \langle (f(M) 1_H - f(A)) x, x \rangle^{1/2} \langle (f(A) - f(m) 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{2} [f(M) - f(m)] \left(\|g(B) x\|^2 - |\langle g(B) x, x \rangle|^2 \right)^{1/2}$$

$$\leq \frac{1}{4} (N - n) [f(M) - f(m)]$$

for any $x \in H, \|x\| = 1.$

Proof. We notice that, since $n := \min_{t \in [m,M]} g(t)$ and $N := \max_{t \in [m,M]} g(t)$, then $n \leq \langle g(B) x, x \rangle \leq N$ which implies that $\langle g(B) x - nx, Mx - g(B) x \rangle \geq 0$ for any $x \in H, \|x\| = 1$. On applying Theorem 223 for $y = Bx, \Gamma = N$ and $\gamma = n$ we deduce the desired result. ∎

Remark 227 *We observe that if the function f takes real values and is a (φ, Φ)-Lipschitzian function on $[m, M]$, then the inequality (3.247) can be improved as follows*

$$|\langle f(A) x, g(B) x \rangle - \langle f(A) x, x \rangle \langle g(B) x, x \rangle| \qquad (3.249)$$

$$\leq \frac{1}{2} (\Phi - \varphi) \int_{m-0}^{M} |\langle E_\lambda x, g(B) x \rangle - \langle E_\lambda x, x \rangle \langle x, g(B) x \rangle| \, d\lambda$$

$$\leq \frac{1}{2} (\Phi - \varphi) \left(\|g(B) x\|^2 - |\langle g(B) x, x \rangle|^2 \right)^{1/2}$$

$$\times \int_{m-0}^{M} (\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle)^{1/2} \, d\lambda$$

$$\leq \frac{1}{2} (\Phi - \varphi) \left(\|g(B) x\|^2 - |\langle g(B) x, x \rangle|^2 \right)^{1/2}$$

$$\times \langle (M 1_H - A) x, x \rangle^{1/2} \langle (A - m 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{4} (M - m) (\Phi - \varphi) \left(\|g(B) x\|^2 - |\langle g(B) x, x \rangle|^2 \right)^{1/2}$$

$$\leq \frac{1}{8} (N - n) (M - m) (\Phi - \varphi)$$

for any $x \in H, \|x\| = 1$.

3.7.4 Applications

By choosing different examples of elementary functions into the above inequalities, one can obtain various Grüss' type inequalities of interest.

For instance, if we choose $f, g : (0, \infty) \to (0, \infty)$ with $f(t) = t^p, g(t) = t^q$ and $p, q > 0$, then for any selfadjoint operators A, B with $Sp(A), Sp(B) \subseteq [m, M] \subset (0, \infty)$ we get from (3.248) the inequalities

$$|\langle A^p x, B^q x \rangle - \langle A^p x, x \rangle \langle B^q x, x \rangle| \qquad (3.250)$$

$$\leq p \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2} \int_{m-0}^{M} (\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle)^{1/2} \lambda^{p-1} d\lambda$$

$$\leq \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2} \langle (M^p 1_H - A^p) x, x \rangle^{1/2} \langle (A^p - m^p 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{2} (M^p - m^p) \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2} \leq \frac{1}{4} (M^q - m^q) (M^p - m^p)$$

for any $x \in H$ with $\|x\| = 1$, where $\{E_\lambda\}_\lambda$ is the spectral family of A.

The same choice of functions considered in the inequality (3.249) produce the result

$$|\langle A^p x, B^q x \rangle - \langle A^p x, x \rangle \langle B^q x, x \rangle| \tag{3.251}$$

$$\leq \frac{1}{2} \Delta_p \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2}$$

$$\times \int_{m-0}^{M} \left(\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \right)^{1/2} d\lambda$$

$$\leq \frac{1}{2} \Delta_p \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2}$$

$$\times \langle (M^p 1_H - A^p) x, x \rangle^{1/2} \langle (A^p - m^p 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{4} (M - m) \Delta_p \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2}$$

$$\leq \frac{1}{8} (M^q - m^q) (M - m) \Delta_p$$

where

$$\Delta_p := p \times \begin{cases} M^{p-1} - m^{p-1} & \text{if } p \geq 1 \\[2mm] \frac{M^{1-p} - m^{1-p}}{M^{1-p} m^{1-p}} & \text{if } 0 < p < 1. \end{cases} \tag{3.252}$$

for any $x \in H$ with $\|x\| = 1$.

Now, if we choose $f(t) = \ln t, t > 0$ and keep the same g then we have the inequalities

$$|\langle \ln A x, B^q x \rangle - \langle \ln A x, x \rangle \langle B^q x, x \rangle| \tag{3.253}$$

$$\leq \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2}$$

$$\times \int_{m-0}^{M} \left(\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \right)^{1/2} \lambda^{-1} d\lambda$$

$$\leq \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2}$$

$$\times \langle (\ln M 1_H - \ln A) x, x \rangle^{1/2} \langle (\ln A - \ln m 1_H) x, x \rangle^{1/2}$$

$$\leq \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2} \ln \sqrt{\frac{M}{m}}$$

$$\leq \frac{1}{2} (M^q - m^q) \ln \sqrt{\frac{M}{m}}$$

and

$$|\langle \ln Ax, B^q x \rangle - \langle \ln Ax, x \rangle \langle B^q x, x \rangle| \qquad (3.254)$$

$$\leq \frac{1}{2} \left(\frac{M-m}{mM} \right) \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2}$$

$$\times \int_{m-0}^{M} \left(\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \right)^{1/2} d\lambda$$

$$\leq \frac{1}{2} \left(\frac{M-m}{mM} \right) \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2}$$

$$\times \langle (M1_H - A) x, x \rangle^{1/2} \langle (A - m1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{4} \frac{(M-m)^2}{mM} \left(\|B^q x\|^2 - |\langle B^q x, x \rangle|^2 \right)^{1/2}$$

$$\leq \frac{1}{8} \left(M^q - m^q \right) \frac{(M-m)^2}{mM}$$

for any $x \in H$ with $\|x\| = 1$.

3.8   Two Operators Grüss' Type Inequalities

3.8.1   Some Representation Results

We start with the following representation result that will play a key role in obtaining various bounds for different choices of functions including continuous functions of bounded variation, Lipschitzian functions or monotonic and continuous functions.

Theorem 228 (Dragomir, 2010, [36]) *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. If $f, g : [m, M] \to \mathbb{C}$ are continuous, then we have the representation*

$$\langle f(A) x, g(B) x \rangle - \langle f(A) x, x \rangle \langle x, g(B) x \rangle \qquad (3.255)$$

$$= \int_{m-0}^{M} \left(\int_{m-0}^{M} \left[\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle \right] d(g(\mu)) \right) d(f(\lambda))$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Integrating by parts in the Riemann-Stieltjes integral and making use of the spectral representation theorem we have

$$\int_{m-0}^{M} \left[\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle \right] df(\lambda) \tag{3.256}$$

$$= \left[\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle \right] f(\lambda) \big|_{m-0}^{M}$$

$$- \int_{m-0}^{M} f(\lambda) \, d \left[\langle E_\lambda x, y \rangle - \langle E_\lambda x, x \rangle \langle x, y \rangle \right]$$

$$= \langle x, y \rangle \int_{m-0}^{M} f(\lambda) \, d \langle E_\lambda x, x \rangle - \int_{m-0}^{M} f(\lambda) \, d \langle E_\lambda x, y \rangle$$

$$= \langle x, y \rangle \langle f(A) x, x \rangle - \langle f(A) x, y \rangle$$

for any $x, y \in H$ with $\|x\| = 1$.

Now, if we chose $y = g(B) x$ in (3.256) then we get that

$$\int_{m-0}^{M} \left[\langle E_\lambda x, g(B) x \rangle - \langle E_\lambda x, x \rangle \langle x, g(B) x \rangle \right] df(\lambda) \tag{3.257}$$

$$= \langle x, g(B) x \rangle \langle f(A) x, x \rangle - \langle f(A) x, g(B) x \rangle$$

for any $x \in H$ with $\|x\| = 1$.

Utilising the spectral representation theorem for B we also have for each fixed $\lambda \in [m, M]$ that

$$\langle E_\lambda x, g(B) x \rangle - \langle E_\lambda x, x \rangle \langle x, g(B) x \rangle \tag{3.258}$$

$$= \left\langle E_\lambda x, \int_{m-0}^{M} g(\mu) \, dF_\mu x \right\rangle - \langle E_\lambda x, x \rangle \left\langle x, \int_{m-0}^{M} g(\mu) \, dF_\mu x \right\rangle$$

$$= \int_{m-0}^{M} g(\mu) \, d \left(\langle E_\lambda x, F_\mu x \rangle \right) - \langle E_\lambda x, x \rangle \int_{m-0}^{M} g(\mu) \, d \left(\langle x, F_\mu x \rangle \right)$$

for any $x \in H$ with $\|x\| = 1$.

Integrating by parts in the Riemann-Stieltjes integral we have

$$\int_{m-0}^{M} g(\mu) \, d \left(\langle E_\lambda x, F_\mu x \rangle \right) = g(\mu) \langle E_\lambda x, F_\mu x \rangle \big|_{m-0}^{M} - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x \rangle \, dg(\mu)$$

$$= g(M) \langle E_\lambda x, x \rangle - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x \rangle \, d(g(\mu))$$

and

$$\int_{m-0}^{M} g(\mu) \, d \left(\langle x, F_\mu x \rangle \right) = g(\mu) \langle x, F_\mu x \rangle \big|_{m-0}^{M} - \int_{m-0}^{M} \langle x, F_\mu x \rangle \, d(g(\mu))$$

$$= g(M) - \int_{m-0}^{M} \langle x, F_\mu x \rangle \, d(g(\mu)),$$

therefore

$$\int_{m-0}^{M} g(\mu) \, d\left(\langle E_\lambda x, F_\mu x\rangle\right) - \langle E_\lambda x, x\rangle \int_{m-0}^{M} g(\mu) \, d\left(\langle x, F_\mu x\rangle\right) \qquad (3.259)$$

$$= g(M) \langle E_\lambda x, x\rangle - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle \, d\left(g(\mu)\right)$$

$$- \langle E_\lambda x, x\rangle \left(g(M) - \int_{m-0}^{M} \langle x, F_\mu x\rangle \, d\left(g(\mu)\right) \right)$$

$$= \langle E_\lambda x, x\rangle \int_{m-0}^{M} \langle x, F_\mu x\rangle \, d\left(g(\mu)\right) - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle \, d\left(g(\mu)\right)$$

$$= \int_{m-0}^{M} \left[\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle\right] d\left(g(\mu)\right)$$

for any $x \in H$ with $\|x\| = 1$ and $\lambda \in [m, M]$.

Utilising (3.257)-(3.259) we deduce the desired result (3.255). ∎

Remark 229 *In particular, if we take $B = A$, then we get from (3.255) the equality*

$$\langle f(A)x, g(A)x\rangle - \langle f(A)x, x\rangle \langle x, g(A)x\rangle \qquad (3.260)$$

$$= \int_{m-0}^{M} \left(\int_{m-0}^{M} \left[\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle\right] d\left(g(\mu)\right) \right) d\left(f(\lambda)\right)$$

for any $x \in H$ with $\|x\| = 1$, which for $g = f$ produces the representation result for the variance of the selfadjoint operator $f(A)$,

$$\|f(A)x\|^2 - \langle f(A)x, x\rangle^2 \qquad (3.261)$$

$$= \int_{m-0}^{M} \left(\int_{m-0}^{M} \left[\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle\right] d\left(f(\mu)\right) \right) d\left(f(\lambda)\right)$$

for any $x \in H$ with $\|x\| = 1$.

3.8.2   Bounds for f of Bounded Variation

The first vectorial Grüss' type inequality when one of the functions is of bounded variation is as follows:

Theorem 230 (Dragomir, 2010, [36]) *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. Also, assume that $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$.*

1. If $g : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then we have the inequality

$$|\langle f(A) x, g(B) x \rangle - \langle f(A) x, x \rangle \langle x, g(B) x \rangle| \tag{3.262}$$

$$\leq \max_{(\lambda, \mu) \in [m, M]^2} |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \bigvee_m^M (g) \bigvee_m^M (f)$$

$$\leq \max_{\lambda \in [m, M]} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2}$$

$$\times \max_{\mu \in [m, M]} [\langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle]^{1/2} \bigvee_m^M (g) \bigvee_m^M (f) \leq \frac{1}{4} \bigvee_m^M (g) \bigvee_m^M (f)$$

for any $x \in H$ with $\|x\| = 1$.

2. If $g : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$ on $[m, M]$, then we have the inequality

$$|\langle f(A) x, g(B) x \rangle - \langle f(A) x, x \rangle \langle x, g(B) x \rangle| \tag{3.263}$$

$$\leq K \max_{\lambda \in [m, M]} \left[\int_{m-0}^M |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \, d\mu \right] \bigvee_m^M (f)$$

$$\leq K \bigvee_m^M (f) \max_{\lambda \in [m, M]} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2}$$

$$\times \int_{m-0}^M [\langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle]^{1/2} \, d\mu$$

$$\leq \frac{1}{2} K \bigvee_m^M (f) \langle (M 1_H - B) x, x \rangle^{1/2} \langle (B - m 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{4} K (M - m) \bigvee_m^M (f)$$

for any $x \in H$ with $\|x\| = 1$.

3. If $g : [m, M] \rightarrow \mathbb{R}$ is continuous and monotonic nondecreasing on $[m, M]$, then we have the inequality

$$|\langle f(A) x, g(B) x \rangle - \langle f(A) x, x \rangle \langle x, g(B) x \rangle| \qquad (3.264)$$

$$\leq \max_{\lambda \in [m, M]} \left[\int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \, dg(\mu) \right] \bigvee_{m}^{M} (f)$$

$$\leq \bigvee_{m}^{M} (f) \max_{\lambda \in [m, M]} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2}$$

$$\times \int_{m-0}^{M} [\langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle]^{1/2} \, dg(\mu)$$

$$\leq \frac{1}{2} \bigvee_{m}^{M} (f) \langle (g(M) 1_H - g(B)) x, x \rangle^{1/2} \langle (g(B) - g(m) 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{4} [g(M) - g(m)] \bigvee_{m}^{M} (f)$$

for any $x \in H$ with $\|x\| = 1$.

Proof. 1. It is well known that if $p : [a, b] \rightarrow \mathbb{C}$ is a continuous function, $v : [a, b] \rightarrow \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \max_{t \in [a, b]} |p(t)| \bigvee_{a}^{b} (v), \qquad (3.265)$$

where $\bigvee_{a}^{b} (v)$ denotes the total variation of v on $[a, b]$.

Now, on utilizing the property (3.265) and the identity (3.255) we have

$$|\langle f(A) x, g(B) x \rangle - \langle f(A) x, x \rangle \langle x, g(B) x \rangle| \qquad (3.266)$$

$$\leq \max_{\lambda \in [m, M]} \left| \int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle] \, d(g(\mu)) \right| \bigvee_{m}^{M} (f)$$

for any $x \in [m, M]$.

The same inequality (3.265) produces the bound

$$
\max_{\lambda \in [m,M]} \left| \int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle] \, d\left(g\left(\mu\right)\right) \right| \tag{3.267}
$$

$$
\leq \max_{\lambda \in [m,M]} \left[\max_{\mu \in [m,M]} |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \right] \bigvee_m^M (f)
$$

$$
= \max_{(\lambda,\mu) \in [m,M]^2} |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \bigvee_m^M (f)
$$

for any $x \in [m, M]$.

By making use of (3.266) and (3.267) we deduce the first part of (3.262).

Further, we notice that by the Schwarz inequality in H we have for any $u, v, e \in H$ with $\|e\| = 1$ that

$$
|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \tag{3.268}
$$

$$
\leq \left(\|u\|^2 - |\langle u, e \rangle|^2 \right)^{1/2} \left(\|v\|^2 - |\langle v, e \rangle|^2 \right)^{1/2}.
$$

Indeed, if we write Schwarz's inequality for the vectors $u - \langle u, e \rangle e$ and $v - \langle v, e \rangle e$ we have

$$
|\langle u - \langle u, e \rangle e, v - \langle v, e \rangle e \rangle| \leq \|u - \langle u, e \rangle e\| \, \|v - \langle v, e \rangle e\|
$$

which, by performing the calculations, is equivalent with (3.268).

Now, on utilizing (3.268), we can state that

$$
|\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \tag{3.269}
$$

$$
\leq \left(\|E_\lambda x\|^2 - |\langle E_\lambda x, x \rangle|^2 \right)^{1/2} \left(\|F_\mu x\|^2 - |\langle F_\mu x, x \rangle|^2 \right)^{1/2}
$$

for any $\lambda, \mu \in [m, M]$.

Since E_λ and F_μ are projections and $E_\lambda, F_\mu \geq 0$ then

$$
\|E_\lambda x\|^2 - |\langle E_\lambda x, x \rangle|^2 = \langle E_\lambda x, x \rangle - \langle E_\lambda x, x \rangle^2 \tag{3.270}
$$

$$
= \langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle \leq \frac{1}{4}
$$

and

$$
\|F_\mu x\|^2 - |\langle F_\mu x, x \rangle|^2 = \langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle \leq \frac{1}{4} \tag{3.271}
$$

for any $\lambda, \mu \in [m, M]$ and $x \in H$ with $\|x\| = 1$.

Now, if we use (3.269)-(3.271) then we get the second part of (3.262).

2. Further, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq L \int_a^b |p(t)| \, dt. \tag{3.272}$$

If we use the inequality (3.272), then we have in the case when g is Lipschitzian with the constant $K > 0$ that

$$\max_{\lambda \in [m, M]} \left| \int_{m-0}^M [\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle] \, d(g(\mu)) \right| \tag{3.273}$$

$$\leq K \max_{\lambda \in [m, M]} \left[\int_{m-0}^M |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \, d\mu \right]$$

for any $x \in H$ with $\|x\| = 1$ and the first part of (3.263) is proved.

Further, by employing (3.269)-(3.271) we also get that

$$\max_{\lambda \in [m, M]} \int_{m-0}^M |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \, d\mu \tag{3.274}$$

$$\leq \max_{\lambda \in [m, M]} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2}$$

$$\times \int_{m-0}^M [\langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle]^{1/2} \, d\mu$$

for any $x \in H$ with $\|x\| = 1$.

If we use the Cauchy-Bunyakovsky-Schwarz integral inequality and the spectral representation theorem, then we have successively

$$\int_{m-0}^M (\langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle)^{1/2} \, d\mu \tag{3.275}$$

$$\leq \left[\int_{m-0}^M \langle F_\mu x, x \rangle \, d\mu \right]^{1/2} \left[\int_{m-0}^M \langle (1_H - F_\mu) x, x \rangle \, d\mu \right]^{1/2}$$

$$= \left[\langle F_\mu x, x \rangle \mu \big|_{m-0}^M - \int_{m-0}^M \mu d \langle F_\mu x, x \rangle \right]^{1/2}$$

$$\times \left[\langle (1_H - F_\mu) x, x \rangle \mu \big|_{m-0}^M - \int_{m-0}^M \mu d \langle (1_H - F_\mu) x, x \rangle \right]$$

$$= \langle (M 1_H - B) x, x \rangle^{1/2} \langle (B - m 1_H) x, x \rangle^{1/2},$$

for any $x \in H$ with $\|x\| = 1$.

On employing now (3.273)-(3.275) we deduce the second part of (3.263). The last part of (3.263) follows by the elementary inequality

$$\alpha\beta \leq \frac{1}{4}\left(\alpha + \beta\right)^2, \alpha\beta \geq 0 \tag{3.276}$$

for the choice $\alpha = \langle (M1_H - B)\, x, x \rangle$ and $\beta = \langle (B - m1_H)\, x, x \rangle$ and the details are omitted.

3. Further, from the theory of Riemann-Stieltjes integral it is also well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p\left(t\right) dv\left(t\right)$ and $\int_a^b \left|p\left(t\right)\right| dv\left(t\right)$ exist and

$$\left|\int_a^b p\left(t\right) dv\left(t\right)\right| \leq \int_a^b \left|p\left(t\right)\right| dv\left(t\right). \tag{3.277}$$

Now, if we assume that g is monotonic nondecreasing on $[m, M]$, then by (3.277) we have that

$$\max_{\lambda \in [m,M]} \left|\int_{m-0}^{M} \left[\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle\right] d\left(g\left(\mu\right)\right)\right| \tag{3.278}$$
$$\leq \max_{\lambda \in [m,M]} \left[\int_{m-0}^{M} \left|\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle\right| dg\left(\mu\right)\right]$$

for any $x \in H$ with $\|x\| = 1$.

Further, by employing (3.269)-(3.271) we also get that

$$\max_{\lambda \in [m,M]} \int_{m-0}^{M} \left|\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle\right| dg\left(\mu\right) \tag{3.279}$$
$$\leq \max_{\lambda \in [m,M]} \left[\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda)\, x, x \rangle\right]^{1/2}$$
$$\times \int_{m-0}^{M} \left[\langle F_\mu x, x \rangle \langle (1_H - F_\mu)\, x, x \rangle\right]^{1/2} dg\left(\mu\right)$$

for any $x \in H$ with $\|x\| = 1$. These prove the first part of (3.264).

If we use the Cauchy-Bunyakovsky-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators and

the spectral representation theorem, then we have successively

$$\int_{m-0}^{M} \left(\langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle \right)^{1/2} dg(\mu) \tag{3.280}$$

$$\leq \left[\int_{m-0}^{M} \langle F_\mu x, x \rangle \, dg(\mu) \right]^{1/2} \left[\int_{m-0}^{M} \langle (1_H - F_\mu) x, x \rangle \, dg(\mu) \right]^{1/2}$$

$$= \left[\langle F_\mu x, x \rangle \, g(\mu) \big|_{m-0}^{M} - \int_{m-0}^{M} g(\mu) \, d \langle F_\mu x, x \rangle \right]^{1/2}$$

$$\times \left[\langle (1_H - F_\mu) x, x \rangle \, g(\mu) \big|_{m-0}^{M} - \int_{m-0}^{M} g(\mu) \, d \langle (1_H - F_\mu) x, x \rangle \right]^{1/2}$$

$$= \langle (g(M) 1_H - g(B)) x, x \rangle^{1/2} \langle (g(B) - g(m) 1_H) x, x \rangle^{1/2},$$

for any $x \in H$ with $\|x\| = 1$.

Utilising (3.280) we then deduce the last part of (3.264). The details are omitted. ∎

Now, in order to provide other results that are similar to the Grüss' type inequalities stated in the introduction, we can state the following corollary:

Corollary 231 (Dragomir, 2010, [36]) *Let A be a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. Also, assume that $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$.*

1. If $g : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then we have the inequality

$$|\langle f(A) x, g(A) x \rangle - \langle f(A) x, x \rangle \langle x, g(A) x \rangle| \tag{3.281}$$

$$\leq \max_{(\lambda,\mu) \in [m,M]^2} |\langle E_\lambda x, x \rangle \langle x, E_\mu x \rangle - \langle E_\lambda x, E_\mu x \rangle| \bigvee_{m}^{M}(g) \bigvee_{m}^{M}(f)$$

$$\leq \max_{\lambda \in [m,M]} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle] \bigvee_{m}^{M}(g) \bigvee_{m}^{M}(f)$$

$$\leq \frac{1}{4} \bigvee_{m}^{M}(g) \bigvee_{m}^{M}(f)$$

for any $x \in H$ with $\|x\| = 1$.

2. *If* $g : [m, M] \rightarrow \mathbb{C}$ *is Lipschitzian with the constant* $K > 0$ *on* $[m, M]$, *then we have the inequality*

$$|\langle f(A) x, g(A) x \rangle - \langle f(A) x, x \rangle \langle x, g(A) x \rangle| \qquad (3.282)$$

$$\leq K \max_{\lambda \in [m,M]} \left[\int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, E_\mu x \rangle - \langle E_\lambda x, E_\mu x \rangle| \, d\mu \right] \bigvee_m^M (f)$$

$$\leq K \bigvee_m^M (f) \max_{\lambda \in [m,M]} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2}$$

$$\times \int_{m-0}^{M} [\langle E_\mu x, x \rangle \langle (1_H - E_\mu) x, x \rangle]^{1/2} \, d\mu$$

$$\leq \frac{1}{2} K \bigvee_m^M (f) \langle (M 1_H - A) x, x \rangle^{1/2} \langle (A - m 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{4} K (M - m) \bigvee_m^M (f)$$

for any $x \in H$ *with* $\|x\| = 1$.

3. *If* $g : [m, M] \rightarrow \mathbb{R}$ *is continuous and monotonic nondecreasing on* $[m, M]$, *then we have the inequality*

$$|\langle f(A) x, g(A) x \rangle - \langle f(A) x, x \rangle \langle x, g(A) x \rangle| \qquad (3.283)$$

$$\leq \max_{\lambda \in [m,M]} \left[\int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, E_\mu x \rangle - \langle E_\lambda x, E_\mu x \rangle| \, dg(\mu) \right] \bigvee_m^M (f)$$

$$\leq \bigvee_m^M (f) \max_{\lambda \in [m,M]} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2}$$

$$\times \int_{m-0}^{M} [\langle E_\mu x, x \rangle \langle (1_H - E_\mu) x, x \rangle]^{1/2} \, dg(\mu)$$

$$\leq \frac{1}{2} \bigvee_m^M (f) \langle (g(M) 1_H - g(A)) x, x \rangle^{1/2} \langle (g(A) - g(m) 1_H) x, x \rangle^{1/2}$$

$$\leq \frac{1}{4} [g(M) - g(m)] \bigvee_m^M (f)$$

for any $x \in H$ *with* $\|x\| = 1$.

Remark 232 *The following inequality for the variance of* $f(A)$ *under the assumptions that* A *is a selfadjoint operators in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$, $\{E_\lambda\}_\lambda$ *is the spectral family of* A *and* $f : [m, M] \rightarrow \mathbb{C}$ *is continuous and of bounded*

variation on $[m, M]$ *can be stated*

$$0 \le \|f(A)x\|^2 - \langle f(A)x, x\rangle^2 \tag{3.284}$$

$$\le \max_{(\lambda,\mu)\in[m,M]^2} |\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle| \left[\bigvee_m^M (f)\right]^2$$

$$\le \max_{\lambda\in[m,M]} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle] \left[\bigvee_m^M (f)\right]^2 \le \frac{1}{4}\left[\bigvee_m^M (f)\right]^2$$

for any $x \in H$ *with* $\|x\| = 1$.

3.8.3   Bounds for f Lipschitzian

The case when the first function is Lipschitzian is as follows:

Theorem 233 (Dragomir, 2010, [36]) *Let* A, B *be two selfadjoint operators in the Hilbert space* H *with the spectra* $Sp(A), Sp(B) \subseteq [m, M]$ *for some real numbers* $m < M$ *and let* $\{E_\lambda\}_\lambda$ *be the spectral family of* A *and* $\{F_\mu\}_\mu$ *the spectral family of* B. *Also, assume that* $f : [m, M] \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on* $[m, M]$.

*   1. If $g : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$ on $[m, M]$, then we have the inequality*

$$|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \tag{3.285}$$

$$\le LK \int_{m-0}^M \int_{m-0}^M |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, d\mu d\lambda$$

$$\le LK \int_{m-0}^M [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle]^{1/2} \, d\lambda$$

$$\times \int_{m-0}^M [\langle F_\mu x, x\rangle \langle (1_H - F_\mu)x, x\rangle]^{1/2} \, d\mu$$

$$\le LK [\langle (M1_H - A)x, x\rangle \langle (A - m1_H)x, x\rangle]^{1/2}$$

$$\times [\langle (M1_H - B)x, x\rangle \langle (B - m1_H)x, x\rangle]^{1/2} \le \frac{1}{4}LK(M - m)^2$$

for any $x \in H$ *with* $\|x\| = 1$.

2. If $g : [m, M] \to \mathbb{R}$ *is continuous and monotonic nondecreasing on* $[m, M]$, *then we have the inequality*

$$|\langle f(A) x, g(B) x\rangle - \langle f(A) x, x\rangle \langle x, g(B) x\rangle| \tag{3.286}$$

$$\leq L \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, dg(\mu) \, d\lambda$$

$$\leq L \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} \, d\lambda$$

$$\times \int_{m-0}^{M} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle]^{1/2} \, dg(\mu)$$

$$\leq L [\langle (M1_H - A) x, x\rangle \langle (A - m1_H) x, x\rangle]^{1/2}$$

$$\times [\langle (g(M) 1_H - g(B)) x, x\rangle \langle (g(B) - g(m) 1_H) x, x\rangle]^{1/2}$$

$$\leq \frac{1}{4} L (M - m) [g(M) - g(m)]$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. 1. We observe that, on utilizing the property (3.272) and the identity (3.255) we have

$$|\langle f(A) x, g(B) x\rangle - \langle f(A) x, x\rangle \langle x, g(B) x\rangle| \tag{3.287}$$

$$\leq L \int_{m-0}^{M} \left| \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle] \, d(g(\mu)) \right| d\lambda$$

for any $x \in H, \|x\| = 1$.

By the same property (3.272) we also have

$$\left| \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle] \, d(g(\mu)) \right| \tag{3.288}$$

$$\leq K \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, d\mu$$

for any $x \in H, \|x\| = 1$ and $\lambda \in [m, M]$.

Therefore, by (3.287) and (3.288) we get

$$|\langle f(A) x, g(B) x\rangle - \langle f(A) x, x\rangle \langle x, g(B) x\rangle| \tag{3.289}$$

$$\leq LK \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, d\mu d\lambda$$

for any $x \in H, \|x\| = 1$, which proves the first inequality in (3.285).

From (3.269)-(3.271) we have

$$|\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \tag{3.290}$$

$$\leq [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle]^{1/2}$$

for any $x \in H, \|x\| = 1$ and $\lambda, \mu \in [m, M]$.

Integrating on $[m, M]^2$ the inequality (3.290) and utilizing the Cauchy-Bunyakowsky-Schwarz integral inequality for the Riemann integral we have

$$\int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \, d\mu d\lambda \qquad (3.291)$$

$$\leq \int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2} \, d\lambda$$

$$\times \int_{m-0}^{M} [\langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle]^{1/2} \, d\mu$$

$$\leq \left[\int_{m-0}^{M} \langle E_\lambda x, x \rangle \, d\lambda \right]^{1/2} \left[\int_{m-0}^{M} \langle (1_H - E_\lambda) x, x \rangle \, d\lambda \right]^{1/2}$$

$$\times \left[\int_{m-0}^{M} \langle F_\mu x, x \rangle \, d\mu \right]^{1/2} \left[\int_{m-0}^{M} \langle (1_H - F_\mu) x, x \rangle \, d\mu \right]^{1/2}.$$

Integrating by parts and utilizing the spectral representation theorem we have

$$\int_{m-0}^{M} \langle E_\lambda x, x \rangle \, d\lambda = \langle E_\lambda x, x \rangle \lambda |_{m-0}^{M} - \int_{m-0}^{M} \lambda d \langle E_\lambda x, x \rangle$$

$$= M - \langle Ax, x \rangle = \langle (M1_H - A) x, x \rangle,$$

$$\int_{m-0}^{M} \langle (1_H - E_\lambda) x, x \rangle \, d\lambda = \langle (A - m1_H) x, x \rangle$$

and the similar equalities for B, providing the second part of (3.285).

The last part follows from (3.276) and we omit the details.

2. Utilising the inequality (3.277) we have

$$\left| \int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle] \, d(g(\mu)) \right| \qquad (3.292)$$

$$\leq \int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \, dg(\mu)$$

which, together with (3.287), produces the inequality

$$|\langle f(A) x, g(B) x \rangle - \langle f(A) x, x \rangle \langle x, g(B) x \rangle| \qquad (3.293)$$

$$\leq L \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \, dg(\mu) \, d\lambda$$

for any $x \in H, \|x\| = 1$.

Now, by utilizing (3.290) and a similar argument to the one outlined above, we deduce the desired result (3.286) and the details are omitted. ∎

The case of one operator is incorporated in

Corollary 234 (Dragomir, 2010, [36]) *Let A be a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. Also, assume that $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$.*

1. If $g : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$ on $[m, M]$, then we have the inequality

$$|\langle f(A) x, g(A) x\rangle - \langle f(A) x, x\rangle \langle x, g(A) x\rangle| \tag{3.294}$$

$$\leq LK \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle| \, d\mu d\lambda$$

$$\leq LK \left(\int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} \, d\lambda \right)^2$$

$$\leq LK \left[\langle (M 1_H - A) x, x\rangle \langle (A - m 1_H) x, x\rangle\right] \leq \frac{1}{4} LK (M - m)^2$$

for any $x \in H$ with $\|x\| = 1$.

2. If $g : [m, M] \to \mathbb{R}$ is continuous and monotonic nondecreasing on $[m, M]$, then we have the inequality

$$|\langle f(A) x, g(A) x\rangle - \langle f(A) x, x\rangle \langle x, g(A) x\rangle| \tag{3.295}$$

$$\leq L \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, dg(\mu) \, d\lambda$$

$$\leq L \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} \, d\lambda$$

$$\times \int_{m-0}^{M} [\langle E_\mu x, x\rangle \langle (1_H - E_\mu) x, x\rangle]^{1/2} \, dg(\mu)$$

$$\leq L \left[\langle (M 1_H - A) x, x\rangle \langle (A - m 1_H) x, x\rangle\right]^{1/2}$$

$$\times \left[\langle (g(M) 1_H - g(A)) x, x\rangle \langle (g(A) - g(m) 1_H) x, x\rangle\right]^{1/2}$$

$$\leq \frac{1}{4} L (M - m) [g(M) - g(m)]$$

for any $x \in H$ with $\|x\| = 1$.

Remark 235 *The following inequality for the variance of $f(A)$ under the assumptions that A is a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ is the spectral family of A and $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant*

L > 0 on [m, M] can be stated

$$0 \leq \|f(A)x\|^2 - \langle f(A)x, x\rangle^2 \tag{3.296}$$

$$\leq L^2 \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle| \, d\mu d\lambda$$

$$\leq L^2 \left(\int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} \, d\lambda \right)^2$$

$$\leq L^2 \left[\langle (M1_H - A) x, x\rangle \langle (A - m1_H) x, x\rangle \right]$$

$$\leq \frac{1}{4} L^2 (M - m)^2$$

for any x ∈ H with ‖x‖ = 1.

3.8.4 Bounds for f Monotonic Nondecreasing

Finally, for the case of two monotonic functions we have the following result as well:

Theorem 236 (Dragomir, 2010, [36]) *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra Sp (A), Sp (B) ⊆ [m, M] for some real numbers m < M and let {E_λ}_λ be the spectral family of A and {F_μ}_μ the spectral family of B. If f, g : [m, M] → ℂ are continuous and monotonic nondecreasing on [m, M], then*

$$|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \tag{3.297}$$

$$\leq \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, dg(\mu) \, df(\lambda)$$

$$\leq \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} \, df(\lambda)$$

$$\times \int_{m-0}^{M} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle]^{1/2} \, dg(\mu)$$

$$\leq [\langle (f(M)1_H - f(A)) x, x\rangle \langle (f(A) - f(m)1_H) x, x\rangle]^{1/2}$$

$$\times [\langle (g(M)1_H - g(B)) x, x\rangle \langle (g(B) - g(m)1_H) x, x\rangle]^{1/2}$$

$$\leq \frac{1}{4} [f(M) - f(m)] [g(M) - g(m)]$$

for any x ∈ H, ‖x‖ = 1.

The details of the proof are omitted.
In particular we have:

Corollary 237 (Dragomir, 2010, [36]) *Let A be a selfadjoint operators in the Hilbert space H with the spectrum Sp (A) ⊆ [m, M] for some real*

numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. If f, g : $[m, M] \to \mathbb{C}$ are continuous and monotonic nondecreasing on $[m, M]$, then

$$|\langle f(A) x, g(A) x \rangle - \langle f(A) x, x \rangle \langle x, g(A) x \rangle| \qquad (3.298)$$

$$\leq \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, E_\mu x \rangle - \langle E_\lambda x, E_\mu x \rangle| \, dg(\mu) \, df(\lambda)$$

$$\leq \int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2} \, df(\lambda)$$

$$\times \int_{m-0}^{M} [\langle E_\mu x, x \rangle \langle (1_H - E_\mu) x, x \rangle]^{1/2} \, dg(\mu)$$

$$\leq [\langle (f(M) 1_H - f(A)) x, x \rangle \langle (f(A) - f(m) 1_H) x, x \rangle]^{1/2}$$

$$\times [\langle (g(M) 1_H - g(A)) x, x \rangle \langle (g(A) - g(m) 1_H) x, x \rangle]^{1/2}$$

$$\leq \frac{1}{4} [f(M) - f(m)] [g(M) - g(m)]$$

for any $x \in H, \|x\| = 1$.

In particular, the following inequality for the variance of $f(A)$ in the case of monotonic nondecreasing functions f holds:

$$0 \leq \|f(A) x\|^2 - \langle f(A) x, x \rangle^2 \qquad (3.299)$$

$$\leq \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, E_\mu x \rangle - \langle E_\lambda x, E_\mu x \rangle| \, df(\mu) \, df(\lambda)$$

$$\leq \left(\int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2} \, df(\lambda) \right)^2$$

$$\leq [\langle (f(M) 1_H - f(A)) x, x \rangle \langle (f(A) - f(m) 1_H) x, x \rangle]$$

$$\leq \frac{1}{4} [f(M) - f(m)]^2$$

for any $x \in H, \|x\| = 1$.

3.8.5   Applications

By choosing different examples of elementary functions into the above inequalities, one can obtain various Grüss' type inequalities of interest.

For instance, if we choose $f, g : (0, \infty) \to (0, \infty)$ with $f(t) = t^p, g(t) = t^q$ and $p, q > 0$, then for any selfadjoint operators A, B with $Sp(A), Sp(B) \subseteq [m, M] \subset (0, \infty)$ we get from (3.297) the inequalities:

$$|\langle A^p x, B^q x \rangle - \langle A^p x, x \rangle \langle B^q x, x \rangle| \tag{3.300}$$

$$\leq pq \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \mu^{q-1} \lambda^{p-1} d\mu d\lambda$$

$$\leq pq \int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2} \lambda^{p-1} d\lambda$$

$$\times \int_{m-0}^{M} [\langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle]^{1/2} \mu^{q-1} d\mu$$

$$\leq [\langle (M^p 1_H - A^p) x, x \rangle \langle (A^p - m^p 1_H) x, x \rangle]^{1/2}$$

$$\times [\langle (M^q 1_H - B^q) x, x \rangle \langle (B^q - m^q 1_H) x, x \rangle]^{1/2}$$

$$\leq \frac{1}{4} (M^p - m^p) (M^q - m^q)$$

for any $x \in H$ with $\|x\| = 1$, where $\{E_\lambda\}_\lambda$ is the spectral family of A and $\{F_\mu\}_\mu$ is the spectral family of B.

When $B = A$ then by the Čebyšev's inequality for functions of same monotonicity the inequality (3.300) becomes

$$0 \leq \langle A^p x, A^q x \rangle - \langle A^p x, x \rangle \langle A^q x, x \rangle \tag{3.301}$$

$$\leq pq \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, E_\mu x \rangle - \langle E_\lambda x, E_\mu x \rangle| \mu^{q-1} \lambda^{p-1} d\mu d\lambda$$

$$\leq pq \int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2} \lambda^{p-1} d\lambda$$

$$\times \int_{m-0}^{M} [\langle E_\mu x, x \rangle \langle (1_H - E_\mu) x, x \rangle]^{1/2} \mu^{q-1} d\mu$$

$$\leq [\langle (M^p 1_H - A^p) x, x \rangle \langle (A^p - m^p 1_H) x, x \rangle]^{1/2}$$

$$\times [\langle (M^q 1_H - B^q) x, x \rangle \langle (B^q - m^q 1_H) x, x \rangle]^{1/2}$$

$$\leq \frac{1}{4} (M^p - m^p) (M^q - m^q)$$

for any $x \in H$ with $\|x\| = 1$ and $p, q > 0$.

Now. define the coefficients

$$\Delta_p := p \times \begin{cases} M^{p-1} - m^{p-1} & \text{if } p \geq 1 \\\\ \frac{M^{1-p} - m^{1-p}}{M^{1-p} m^{1-p}} & \text{if } 0 < p < 1. \end{cases} \tag{3.302}$$

On utilizing the inequality (3.285) for the same power functions considered above, we can state the inequality

$$|\langle A^p x, B^q x \rangle - \langle A^p x, x \rangle \langle B^q x, x \rangle| \tag{3.303}$$

$$\leq \Delta_p \Delta_q \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, F_\mu x \rangle - \langle E_\lambda x, F_\mu x \rangle| \, d\mu d\lambda$$

$$\leq \Delta_p \Delta_q \int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2} \, d\lambda$$

$$\times \int_{m-0}^{M} [\langle F_\mu x, x \rangle \langle (1_H - F_\mu) x, x \rangle]^{1/2} \, d\mu$$

$$\leq \Delta_p \Delta_q [\langle (M1_H - A) x, x \rangle \langle (A - m1_H) x, x \rangle]^{1/2}$$

$$\times [\langle (M1_H - B) x, x \rangle \langle (B - m1_H) x, x \rangle]^{1/2} \leq \frac{1}{4} \Delta_p \Delta_q (M - m)^2$$

for any $x \in H$ with $\|x\| = 1$ and $p, q > 0$.

In particular, for $B = A$ we have

$$0 \leq \langle A^p x, A^q x \rangle - \langle A^p x, x \rangle \langle A^q x, x \rangle \tag{3.304}$$

$$\leq \Delta_p \Delta_q \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x \rangle \langle x, E_\mu x \rangle - \langle E_\lambda x, E_\mu x \rangle| \, d\mu d\lambda$$

$$\leq \Delta_p \Delta_q \left(\int_{m-0}^{M} [\langle E_\lambda x, x \rangle \langle (1_H - E_\lambda) x, x \rangle]^{1/2} \, d\lambda \right)^2$$

$$\leq \Delta_p \Delta_q [\langle (M1_H - A) x, x \rangle \langle (A - m1_H) x, x \rangle] \leq \frac{1}{4} \Delta_p \Delta_q (M - m)^2$$

for any $x \in H$ with $\|x\| = 1$ and $p, q > 0$.

Similar results can be stated if $p < 0$ or $q < 0$. However the details are left to the interest reader.

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4
Inequalities of Ostrowski Type

4.1 Introduction

Ostrowski's type inequalities provide sharp error estimates in approximating the value of a function by its integral mean. They can be utilized to obtain a priory error bounds for different quadrature rules in approximating the Riemann integral by different Riemann sums. They also shows, in general, that the mid-point rule provides the best approximation in the class of all Riemann sums sampled in the interior points of a given partition.

As revealed by a simple search in the data base *MathSciNet* of the *American Mathematical Society* with the key words "Ostrowski" and "inequality" in the title, an exponential evolution of research papers devoted to this result has been registered in the last decade. There are now at least 280 papers that can be found by performing the above search. Numerous extensions, generalisations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Probability Theory and other fields have been also given.

In the present chapter we present some recent results obtained by the author in extending Ostrowski inequality in various directions for continuous functions of selfadjoint operators in complex Hilbert spaces. As far as we know, the obtained results are new with no previous similar results ever obtained in the literature.

Applications for mid-point inequalities and some elementary functions of operators such as the power function, the logarithmic and exponential functions are provided as well.

4.2   Scalar Ostrowski's Type Inequalities

In the scalar case, comparison between functions and integral means are incorporated in Ostrowski type inequalities as mentioned below.

The first result in this direction is known in the literature as Ostrowski's inequality [44].

Theorem 238 *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M \qquad (4.1)$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity..

The following Ostrowski type result for absolutely continuous functions holds (see [34] − [36]).

Theorem 239 *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. Then, for all $x \in [a, b]$, we have:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \qquad\qquad\qquad\qquad (4.2)$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty & if \quad f' \in L_\infty [a, b]; \\[3ex] \frac{1}{(p+1)^{\frac{1}{p}}} \left[\left(\frac{x-a}{b-a} \right)^{p+1} + \left(\frac{b-x}{b-a} \right)^{p+1} \right]^{\frac{1}{p}} (b-a)^{\frac{1}{p}} \|f'\|_q & if \quad f' \in L_q [a, b], \\ & \qquad \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; \\[3ex] \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \|f'\|_1 ; & \end{cases}$$

where $\|\cdot\|_r$ $(r \in [1, \infty])$ are the usual Lebesgue norms on $L_r [a, b]$, i.e.,

$$\|g\|_\infty := \operatorname*{ess\ sup}_{t \in [a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\, dt \right)^{\frac{1}{r}}, \quad r \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 238.

The above inequalities can also be obtained from the Fink result in [39] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see for instance [32] and the references therein for earlier contributions):

Theorem 240 *Let $f : [a, b] \to \mathbb{R}$ be of $r - H -$Hölder type, i.e.,*

$$|f(x) - f(y)| \leq H |x - y|^r, \ \text{for all} \ x, y \in [a, b], \qquad (4.3)$$

where $r \in (0, 1]$ and $H > 0$ are fixed. Then, for all $x \in [a, b]$, we have the inequality:

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t)\, dt \right| \qquad (4.4)$$

$$\leq \frac{H}{r + 1} \left[\left(\frac{b - x}{b - a} \right)^{r+1} + \left(\frac{x - a}{b - a} \right)^{r+1} \right] (b - a)^r.$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [24])

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b - a} \right)^2 \right] (b - a) L. \qquad (4.5)$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [23]).

Theorem 241 *Assume that $f : [a, b] \to \mathbb{R}$ is of bounded variation and denote by $\overset{b}{\underset{a}{\bigvee}}(f)$ its total variation. Then*

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] \overset{b}{\underset{a}{\bigvee}}(f) \qquad (4.6)$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, i.e., f is monotonically increasing, then the inequality (4.6) may be improved in the following manner [12] (see also the monograph [33]).

Theorem 242 *Let* $f : [a, b] \rightarrow \mathbb{R}$ *be monotonic nondecreasing. Then for all* $x \in [a, b]$, *we have the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \tag{4.7}$$

$$\leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b sgn(t-x) f(t) \, dt \right\}$$

$$\leq \frac{1}{b-a} \left\{ (x-a) [f(x) - f(a)] + (b-x) [f(b) - f(x)] \right\}$$

$$\leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)].$$

All the inequalities in (4.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

For other scalar Ostrowski's type inequalities, see [2]-[4] and [25].

4.3 Ostrowski's type Inequalities for Hölder Continuous Functions

4.3.1 Introduction

Let U be a selfadjoint operator on the Hilbert space $(H, \langle., .\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be its *spectral family*. Then for any continuous function $f : [m, M] \rightarrow \mathbb{C}$, it is well known that we have the following *spectral representation* theorem in terms of the *Riemann-Stieltjes integral*:

$$\langle f(U) x, x \rangle = \int_{m-0}^{M} f(\lambda) \, d(\langle E_\lambda x, x \rangle), \tag{4.8}$$

for any $x \in H$ with $\|x\| = 1$. The function $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* on the interval $[m, M]$ and

$$g_x(m-0) = 0 \text{ and } g_x(M) = 1 \tag{4.9}$$

for any $x \in H$ with $\|x\| = 1$.

Utilising the representation (4.8) and the following Ostrowski's type inequality for the Riemann-Stieltjes integral obtained by the author in [28]:

$$\left| f(s) \left[u(b) - u(a) \right] - \int_a^b f(t) \, du(t) \right| \tag{4.10}$$

$$\leq L \left[\frac{1}{2} (b - a) + \left| s - \frac{a+b}{2} \right| \right]^r \bigvee_a^b (u)$$

for any $s \in [a, b]$, provided that f is of $r-L-$Hölder type on $[a, b]$ (see (4.11) below), u is of *bounded variation* on $[a, b]$ and $\bigvee_a^b (u)$ denotes the *total variation* of u on $[a, b]$, we obtained the following inequality of Ostrowski type for selfadjoint operators:

Theorem 243 (Dragomir, 2008, [29]) *Let A and B be selfadjoint operators with $Sp(A)$, $Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type, i.e., for a given $r \in (0, 1]$ and $L > 0$ we have*

$$|f(s) - f(t)| \leq L |s - t|^r \text{ for any } s, t \in [m, M], \tag{4.11}$$

then we have the inequality:

$$|f(s) - \langle f(A) x, x \rangle| \leq L \left[\frac{1}{2} (M - m) + \left| s - \frac{m+M}{2} \right| \right]^r, \tag{4.12}$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.
   Moreover, we have

$$|\langle f(B) y, y \rangle - \langle f(A) x, x \rangle| \tag{4.13}$$
$$\leq \langle |f(B) - \langle f(A) x, x \rangle \cdot 1_H | y, y \rangle$$
$$\leq L \left[\frac{1}{2} (M - m) + \left\langle \left| B - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle \right]^r,$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

With the above assumptions for f, A and B we have the following particular inequalities of interest:

$$\left| f \left(\frac{m+M}{2} \right) - \langle f(A) x, x \rangle \right| \leq \frac{1}{2^r} L (M - m)^r \tag{4.14}$$

and

$$|f(\langle Ax, x \rangle) - \langle f(A) x, x \rangle| \tag{4.15}$$
$$\leq L \left[\frac{1}{2} (M - m) + \left| \langle Ax, x \rangle - \frac{m+M}{2} \right| \right]^r,$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequalities:

$$
|\langle f(A) y, y \rangle - \langle f(A) x, x \rangle| \tag{4.16}
$$
$$
\leq \langle |f(A) - \langle f(A) x, x \rangle \cdot 1_H| \, y, y \rangle
$$
$$
\leq L \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle \right]^r,
$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$,

$$
|\langle [f(B) - f(A)] x, x \rangle| \tag{4.17}
$$
$$
\leq \langle |f(B) - \langle f(A) x, x \rangle \cdot 1_H| \, x, x \rangle
$$
$$
\leq L \left[\frac{1}{2} (M - m) + \left\langle \left| B - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle \right]^r
$$

and, more particularly,

$$
\langle |f(A) - \langle f(A) x, x \rangle \cdot 1_H| \, x, x \rangle \tag{4.18}
$$
$$
\leq L \left[\frac{1}{2} (M - m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle \right]^r,
$$

for any $x \in H$ with $\|x\| = 1$.

We also have the norm inequality

$$
\|f(B) - f(A)\| \leq L \left[\frac{1}{2} (M - m) + \left\| B - \frac{m+M}{2} \cdot 1_H \right\| \right]^r. \tag{4.19}
$$

For various generalizations, extensions and related Ostrowski type inequalities for functions of one or several variables see the monograph [31] and the references therein.

4.3.2 More Inequalities of Ostrowski's Type

The following result holds:

Theorem 244 (Dragomir, 2010, [30]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L -$Hölder type with $r \in (0, 1]$, then we have the inequality:*

$$
|f(s) - \langle f(A) x, x \rangle| \leq L \langle |s \cdot 1_H - A| x, x \rangle^r \tag{4.20}
$$
$$
\leq L \left[(s - \langle Ax, x \rangle)^2 + D^2(A; x) \right]^{r/2},
$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$, where $D(A; x)$ is the variance of the selfadjoint operator A in x and is defined by

$$
D(A; x) := \left(\|Ax\|^2 - \langle Ax, x \rangle^2 \right)^{1/2},
$$

where $x \in H$ with $\|x\| = 1$.

Proof. First of all, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [40, p. 5]) applied for the modulus, we can state that

$$|\langle h\,(A)\,x, x\rangle| \leq \langle |h\,(A)|\,x, x\rangle \qquad (M)$$

for any $x \in H$ with $\|x\| = 1$, where h is a continuous function on $[m, M]$.

Utilising the property (M) we then get

$$|f\,(s) - \langle f\,(A)\,x, x\rangle| = |\langle f\,(s) \cdot 1_H - f\,(A)\,x, x\rangle| \qquad (4.21)$$
$$\leq \langle |f\,(s) \cdot 1_H - f\,(A)|\,x, x\rangle$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m, M]$.

Since f is of $r - L-$Hölder type, then for any $t, s \in [m, M]$ we have

$$|f\,(s) - f\,(t)| \leq L\,|s - t|^r. \qquad (4.22)$$

If we fix $s \in [m, M]$ and apply the property (P) for the inequality (4.22) and the operator A we get

$$\langle |f\,(s) \cdot 1_H - f\,(A)|\,x, x\rangle \leq L\,\langle |s \cdot 1_H - A|^r\,x, x\rangle \qquad (4.23)$$
$$\leq L\,\langle |s \cdot 1_H - A|\,x, x\rangle^r$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m, M]$, where, for the last inequality we have used the fact that if P is a positive operator and $r \in (0, 1)$ then, by the Hölder-McCarthy inequality [42],

$$\langle P^r x, x\rangle \leq \langle Px, x\rangle^r \qquad (HM)$$

for any $x \in H$ with $\|x\| = 1$. This proves the fist inequality in (4.20).

Now, observe that for any bounded linear operator T we have

$$\langle |T|\,x, x\rangle = \left\langle (T^*T)^{1/2}\,x, x\right\rangle \leq \langle (T^*T)\,x, x\rangle^{1/2} = \|Tx\|$$

for any $x \in H$ with $\|x\| = 1$ which implies that

$$\langle |s \cdot 1_H - A|\,x, x\rangle^r \leq \|sx - Ax\|^r \qquad (4.24)$$
$$= \left(s^2 - 2s\,\langle Ax, x\rangle + \|Ax\|^2\right)^{r/2}$$
$$= \left[(s - \langle Ax, x\rangle)^2 + \|Ax\|^2 - \langle Ax, x\rangle^2\right]^{r/2}$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m, M]$.

Finally, on making use of (4.21), (4.23) and (4.24) we deduce the desired result (4.20). ∎

Remark 245 *If we choose in (4.20) $s = \frac{m+M}{2}$, then we get the sequence of inequalities*

$$\left| f\left(\frac{m+M}{2}\right) - \langle f(A)x, x\rangle \right| \tag{4.25}$$

$$\le L\left\langle \left| \frac{m+M}{2} \cdot 1_H - A\right| x, x\right\rangle^r$$

$$\le L\left[\left(\frac{m+M}{2} - \langle Ax, x\rangle\right)^2 + D^2(A;x) \right]^{r/2}$$

$$\le L\left[\frac{1}{4}(M-m)^2 + D^2(A;x) \right]^{r/2} \le \frac{1}{2^r}L(M-m)^r$$

for any $x \in H$ with $\|x\| = 1$, since, obviously,

$$\left(\frac{m+M}{2} - \langle Ax, x\rangle\right)^2 \le \frac{1}{4}(M-m)^2$$

and

$$D^2(A;x) \le \frac{1}{4}(M-m)^2$$

for any $x \in H$ with $\|x\| = 1$.

We notice that the inequality (4.25) provides a refinement for the result (4.14) above.

The best inequality we can get from (4.20) is incorporated in the following:

Corollary 246 (Dragomir, 2010, [30]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type with $r \in (0, 1]$, then we have the inequality*

$$|f(\langle Ax, x\rangle) - \langle f(A)x, x\rangle| \le L\langle |\langle Ax, x\rangle \cdot 1_H - A| x, x\rangle^r \tag{4.26}$$
$$\le LD^r(A;x),$$

for any $x \in H$ with $\|x\| = 1$.

The inequality (4.20) may be used to obtain other inequalities for two selfadjoint operators as follows:

Corollary 247 (Dragomir, 2010, [30]) *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type with $r \in (0, 1]$, then we have the inequality*

$$|\langle f(B)y, y\rangle - \langle f(A)x, x\rangle| \tag{4.27}$$

$$\le L\left[(\langle By, y\rangle - \langle Ax, x\rangle)^2 + D^2(A;x) + D^2(B;y) \right]^{r/2}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. If we apply the property (P) to the inequality (4.20) and for the operator B, then we get

$$\langle |f(B) - \langle f(A) x, x \rangle \cdot 1_H | y, y \rangle \tag{4.28}$$

$$\leq L \left\langle \left[(B - \langle Ax, x \rangle \cdot 1_H)^2 + D^2(A; x) \cdot 1_H \right]^{r/2} y, y \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Utilising the inequality (M) we also have that

$$|f(\langle By, y \rangle) - \langle f(A) x, x \rangle| \leq \langle |f(B) - \langle f(A) x, x \rangle \cdot 1_H | y, y \rangle \tag{4.29}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, by the Hölder-McCarthy inequality (HM) we also have

$$\left\langle \left[(B - \langle Ax, x \rangle \cdot 1_H)^2 + D^2(A; x) \cdot 1_H \right]^{r/2} y, y \right\rangle \tag{4.30}$$

$$\leq \left\langle \left[(B - \langle Ax, x \rangle \cdot 1_H)^2 + D^2(A; x) \cdot 1_H \right] y, y \right\rangle^{r/2}$$

$$= \left((\langle By, y \rangle - \langle Ax, x \rangle)^2 + D^2(A; x) + D^2(B; y) \right)^{r/2}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

On making use of (4.28)-(4.30) we deduce the desired result (4.27). ∎

Remark 248 *Since*

$$D^2(A; x) \leq \frac{1}{4}(M - m)^2, \tag{4.31}$$

then we obtain from (4.27) the following vector inequalities

$$|\langle f(A) y, y \rangle - \langle f(A) x, x \rangle| \tag{4.32}$$

$$\leq L \left[(\langle Ay, y \rangle - \langle Ax, x \rangle)^2 + D^2(A; x) + D^2(A; y) \right]^{r/2}$$

$$\leq L \left[(\langle Ay, y \rangle - \langle Ax, x \rangle)^2 + \frac{1}{2}(M - m)^2 \right]^{r/2},$$

and

$$|\langle [f(B) - f(A)] x, x \rangle| \tag{4.33}$$

$$\leq L \left[\langle (B - A) x, x \rangle^2 + D^2(A; x) + D^2(B; x) \right]^{r/2}$$

$$\leq L \left[\langle (B - A) x, x \rangle^2 + \frac{1}{2}(M - m)^2 \right]^{r/2}.$$

In particular, we have the norm inequality

$$\|f(B) - f(A)\| \leq L \left[\|B - A\|^2 + \frac{1}{2}(M - m)^2 \right]^{r/2}. \tag{4.34}$$

The following result provides convenient examples for applications:

Corollary 249 (Dragomir, 2010, [30]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is absolutely continuous on $[m, M]$, then we have the inequality:*

$$|f(s) - \langle f(A)x, x\rangle| \qquad\qquad (4.35)$$

$$\leq \begin{cases} \langle |s \cdot 1_H - A| x, x\rangle \, \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m, M], \\[2ex] \langle |s \cdot 1_H - A| x, x\rangle^{1/q} \, \|f'\|_{[m,M],p} & \begin{array}{l} \text{if } f' \in L_p[m, M], \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{array} \end{cases}$$

$$\leq \begin{cases} \left[(s - \langle Ax, x\rangle)^2 + D^2(A; x) \right]^{1/2} \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m, M], \\[2ex] \left[(s - \langle Ax, x\rangle)^2 + D^2(A; x) \right]^{\frac{1}{2q}} \|f'\|_{[m,M],p} & \begin{array}{l} \text{if } f' \in L_p[m, M], \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{array} \end{cases}$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$, where $\|f'\|_{[m,M],\ell}$ are the Lebesgue norms, i.e.,

$$\|f'\|_{[m,M],\ell} := \begin{cases} ess \sup_{t \in [m,M]} |f'(t)| & \text{if } \ell = \infty \\[2ex] \left(\int_m^M |f'(t)|^p \, dt \right)^{1/p} & \text{if } \ell = p \geq 1. \end{cases}$$

Proof. Follows from Theorem 244 and on tacking into account that if $f : [m, M] \longrightarrow \mathbb{R}$ is absolutely continuous on $[m, M]$, then for any $s, t \in [m, M]$ we have

$$|f(s) - f(t)|$$
$$= \left| \int_t^s f'(u) \, du \right|$$
$$\leq \begin{cases} |s - t| \, ess \sup_{t \in [m,M]} |f'(t)| & \text{if } f' \in L_\infty[m, M] \\[2ex] |s - t|^{1/q} \left(\int_m^M |f'(t)|^p \, dt \right)^{1/p} & \text{if } f' \in L_p[m, M], p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

∎

Remark 250 *It is clear that all the inequalities from Corollaries 246, 247 and Remark 248 may be stated for absolutely continuous functions. How-*

ever, we mention here only one, namely

$$|f(\langle Ax, x\rangle) - \langle f(A)x, x\rangle| \qquad (4.36)$$

$$\leq \begin{cases} \langle|\langle Ax, x\rangle \cdot 1_H - A|x, x\rangle \, \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m, M] \\[2ex] \langle|\langle Ax, x\rangle \cdot 1_H - A|x, x\rangle^{1/q} \, \|f'\|_{[m,M],p} & \begin{array}{l} \text{if } f' \in L_p[m, M], \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{array} \end{cases}$$

$$\leq \begin{cases} D(A; x) \, \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m, M] \\[2ex] D^{1/q}(A; x) \, \|f'\|_{[m,M],p} & \begin{array}{l} \text{if } f' \in L_p[m, M], \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{array} \end{cases}$$

4.3.3   The Case of $(\varphi, \Phi) -$ *Lipschitzian Functions*

The following result can be stated:

Proposition 251 (Dragomir, 2010, [30]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is $(\gamma, \Gamma) -$Lipschitzian on $[m, M]$, then we have the inequality*

$$|f(\langle Ax, x\rangle) - \langle f(A)x, x\rangle| \leq \frac{1}{2}(\Gamma - \gamma) \langle|\langle Ax, x\rangle \cdot 1_H - A|x, x\rangle \qquad (4.37)$$

$$\leq \frac{1}{2}(\Gamma - \gamma) D(A; x),$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Follows by Corollary 246 on taking into account that in this case we have $r = 1$ and $L = \frac{1}{2}(\Gamma - \gamma)$. ∎

We can use the result (4.37) for the particular case of convex functions to provide an interesting reverse inequality for the Jensen's type operator inequality due to Mond and Pečarić [43] (see also [40, p. 5]):

Theorem 252 (Mond-Pečarić, 1993, [43]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \qquad (\text{MP})$$

for each $x \in H$ with $\|x\| = 1$.

Corollary 253 (Dragomir, 2010, [30]) *With the assumptions of Theorem 252 we have the inequality*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \qquad (4.38)$$

$$\leq \frac{1}{2}\left(f'_-(M) - f'_+(m)\right)\langle|\langle Ax, x\rangle \cdot 1_H - A|x, x\rangle$$

$$\leq \frac{1}{2}\left(f'_-(M) - f'_+(m)\right) D(A; x) \leq \frac{1}{4}\left(f'_-(M) - f'_+(m)\right)(M - m)$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. Follows by Proposition 251 on taking into account that

$$f'_+ (m) (t - s) \le f (t) - f (s) \le f'_- (M) (t - s)$$

for each s, t with the property that $M > t > s > m$. ∎
    The following result may be stated as well:

Proposition 254 (Dragomir, 2010, [30]) *Let A be a selfadjoint opera-tor with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is $(\gamma, \Gamma) - Lipschitzian$ on $[m, M]$, then we have the inequality*

$$|f (\langle Ax, x \rangle) - \langle f (A) x, x \rangle| \qquad (4.39)$$
$$\le \frac{1}{2} (\Gamma - \gamma) \left[\frac{1}{2} (M - m) + \left| \langle Ax, x \rangle - \frac{m + M}{2} \right| \right]$$

for any $x \in H$ *with* $\|x\| = 1$.

    The following particular case for convex functions holds:

Corollary 255 (Dragomir, 2010, [30]) *With the assumptions of Theo-rem 252 we have the inequality*

$$(0 \le) \langle f (A) x, x \rangle - f (\langle Ax, x \rangle) \qquad (4.40)$$
$$\le \frac{1}{2} (f'_- (M) - f'_+ (m)) \left[\frac{1}{2} (M - m) + \left| \langle Ax, x \rangle - \frac{m + M}{2} \right| \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

4.3.4 Related Results

In the previous sections we have compared amongst other the following quantities

$$f \left(\frac{m + M}{2} \right) \text{ and } f (\langle Ax, x \rangle)$$

with $\langle f (A) x, x \rangle$ for a selfadjoint operator A on the Hilbert space H with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $f : [m, M] \longrightarrow \mathbb{R}$ a function of $r - L - $ Hölder type with $r \in (0, 1]$ and $x \in H$ with $\|x\| = 1$.
    Since, obviously,

$$m \le \frac{1}{M - m} \int_m^M f (t) \, dt \le M,$$

then is also natural to compare $\frac{1}{M-m} \int_m^M f (t) \, dt$ with $\langle f (A) x, x \rangle$ under the same assumptions for f, A and x.
    The following result holds:

Theorem 256 (Dragomir, 2010, [30]) *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type with $r \in (0, 1]$, then we have the inequality:*

$$\left| \frac{1}{M - m} \int_m^M f(s)\, dt - \langle f(A)\, x, x \rangle \right| \tag{4.41}$$

$$\leq \frac{1}{r + 1} L (M - m)^r$$

$$\times \left[\left\langle \left(\frac{M \cdot 1_H - A}{M - m} \right)^{r+1} x, x \right\rangle + \left\langle \left(\frac{A - m \cdot 1_H}{M - m} \right)^{r+1} x, x \right\rangle \right]$$

$$\leq \frac{1}{r + 1} L (M - m)^r,$$

for any $x \in H$ with $\|x\| = 1$.

In particular, if $f : [m, M] \longrightarrow \mathbb{R}$ is Lipschitzian with a constant K, then

$$\left| \frac{1}{M - m} \int_m^M f(s)\, dt - \langle f(A)\, x, x \rangle \right| \tag{4.42}$$

$$\leq K (M - m) \left[\frac{1}{4} + \frac{1}{(M - m)^2} \left(D^2 (A; x) + \left(\langle Ax, x \rangle - \frac{m + M}{2} \right)^2 \right) \right]$$

$$\leq \frac{1}{2} K (M - m)$$

for any $x \in H$ with $\|x\| = 1$.

Proof. We use the following Ostrowski's type result (see for instance [31, p. 3]) written for the function f that is of $r - L-$Hölder type on the interval $[m, M]$:

$$\left| \frac{1}{M - m} \int_m^M f(s)\, dt - f(t) \right| \tag{4.43}$$

$$\leq \frac{L}{r + 1} (M - m)^r \left[\left(\frac{M - t}{M - m} \right)^{r+1} + \left(\frac{t - m}{M - m} \right)^{r+1} \right]$$

for any $t \in [m, M]$.

If we apply the properties (P) and (M) then we have successively

$$\left| \frac{1}{M-m} \int_m^M f(s)\, dt - \langle f(A) x, x \rangle \right| \tag{4.44}$$

$$\leq \left\langle \left| \frac{1}{M-m} \int_m^M f(s)\, dt - f(A) \right| x, x \right\rangle$$

$$\leq \frac{L}{r+1} (M-m)^r$$

$$\times \left[\left\langle \left(\frac{M \cdot 1_H - A}{M-m} \right)^{r+1} x, x \right\rangle + \left\langle \left(\frac{A - m \cdot 1_H}{M-m} \right)^{r+1} x, x \right\rangle \right]$$

which proves the first inequality in (4.41).

Utilising the Lah-Ribarić inequality version for selfadjoint operators A with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and convex functions $g : [m, M] \to \mathbb{R}$, namely (see for instance [40, p. 57]):

$$\langle g(A) x, x \rangle \leq \frac{M - \langle Ax, x \rangle}{M - m} g(m) + \frac{\langle Ax, x \rangle - m}{M - m} g(M)$$

for any $x \in H$ with $\|x\| = 1$, then we get for the convex function $g(t) := \left(\frac{M-t}{M-m} \right)^{r+1}$,

$$\left\langle \left(\frac{M \cdot 1_H - A}{M-m} \right)^{r+1} x, x \right\rangle \leq \frac{M - \langle Ax, x \rangle}{M - m}$$

and for the convex function $g(t) := \left(\frac{t-m}{M-m} \right)^{r+1}$,

$$\left\langle \left(\frac{A - m \cdot 1_H}{M-m} \right)^{r+1} x, x \right\rangle \leq \frac{\langle Ax, x \rangle - m}{M - m}$$

for any $x \in H$ with $\|x\| = 1$.

Now, on making use of the last two inequalities, we deduce the second part of (4.41).

Since

$$\frac{1}{2} \left\langle \left(\frac{M \cdot 1_H - A}{M-m} \right)^2 x, x \right\rangle + \left\langle \left(\frac{A - m \cdot 1_H}{M-m} \right)^2 x, x \right\rangle$$

$$= \frac{1}{4} + \frac{1}{(M-m)^2} \left(D^2(A; x) + \left(\langle Ax, x \rangle - \frac{m+M}{2} \right)^2 \right)$$

for any $x \in H$ with $\|x\| = 1$, then on choosing $r = 1$ in (4.41) we deduce the desired result (4.42). ∎

Remark 257 *We should notice from the proof of the above theorem, we also have the following inequalities in the operator order of* $B(H)$

$$\left| f(A) - \left(\frac{1}{M-m} \int_m^M f(s)\, dt \right) \cdot 1_H \right| \tag{4.45}$$

$$\leq \frac{L}{r+1} (M-m)^r \left[\left(\frac{M \cdot 1_H - A}{M-m} \right)^{r+1} + \left(\frac{A - m \cdot 1_H}{M-m} \right)^{r+1} \right]$$

$$\leq \frac{1}{r+1} L (M-m)^r \cdot 1_H.$$

The following particular case is of interest:

Corollary 258 (Dragomir, 2010, [30]) *Let* A *be a selfadjoint operator with* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is* $(\gamma, \Gamma)-Lipschitzian$ *on* $[m, M]$, *then we have the inequality*

$$\left| \langle f(A) x, x \rangle - \frac{\Gamma+\gamma}{2} - \frac{1}{M-m} \int_m^M f(s)\, dt + \frac{\Gamma+\gamma}{2} \cdot \frac{m+M}{2} \right| \tag{4.46}$$

$$\leq \frac{1}{2} (\Gamma - \gamma)(M-m)$$

$$\times \left[\frac{1}{4} + \frac{1}{(M-m)^2} \left(D^2(A; x) + \left(\langle Ax, x \rangle - \frac{m+M}{2} \right)^2 \right) \right]$$

$$\leq \frac{1}{4} (\Gamma - \gamma)(M-m).$$

Proof. Follows by (4.42) applied for the $\frac{1}{2}(\Gamma - \gamma)$-Lipschitzian function $f - \frac{\Gamma+\gamma}{2} \cdot e.$ ∎

4.3.5   Applications for Some Particular Functions

1. We have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality:

Theorem 259 (Hölder-McCarthy, 1967, [42]) *Let* A *be a selfadjoint positive operator on a Hilbert space* H. *Then*
  (i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ *for all* $r > 1$ *and* $x \in H$ *with* $\|x\| = 1$;
  (ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ *for all* $0 < r < 1$ *and* $x \in H$ *with* $\|x\| = 1$;
  (iii) *If* A *is invertible, then* $\langle A^{-r} x, x \rangle \geq \langle Ax, x \rangle^{-r}$ *for all* $r > 0$ *and* $x \in H$ *with* $\|x\| = 1$.

We can provide the following reverse inequalities:

Proposition 260 *Let* A *be a selfadjoint positive operator on a Hilbert space* H *and* $0 < r < 1$. *Then*

$$(0 \leq) \langle Ax, x \rangle^r - \langle A^r x, x \rangle \leq \langle |\langle Ax, x \rangle \cdot 1_H - A| x, x \rangle^r \leq D^r(A; x) \tag{4.47}$$

for all $x \in H$ *with* $\|x\| = 1$.

Proof. Follows from Corollary 246 by taking into account that the function $f(t) = t^r$ is of $r - L-$Hölder type with $L = 1$ on any compact interval of $(0, \infty)$. ∎

On making use of Corollary 253 we can state the following result as well:

Proposition 261 *Let A be a selfadjoint positive operator on a Hilbert space H. Assume that $Sp(A) \subseteq [m, M] \subseteq [0, \infty)$.*
(i) We have

$$0 \le \langle A^r x, x \rangle - \langle Ax, x \rangle^r \tag{4.48}$$

$$\le \frac{1}{2} r \left(M^{r-1} - m^{r-1} \right) \langle |\langle Ax, x \rangle \cdot 1_H - A| x, x \rangle$$

$$\le \frac{1}{2} r \left(M^{r-1} - m^{r-1} \right) D(A; x) \le \frac{1}{4} r \left(M^{r-1} - m^{r-1} \right) (M - m)$$

for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
(ii) We also have

$$0 \le \langle Ax, x \rangle^r - \langle A^r x, x \rangle \tag{4.49}$$

$$\le \frac{1}{2} r \left(\frac{M^{1-r} - m^{1-r}}{m^{1-r} M^{1-r}} \right) \langle |\langle Ax, x \rangle \cdot 1_H - A| x, x \rangle$$

$$\le \frac{1}{2} r \left(\frac{M^{1-r} - m^{1-r}}{m^{1-r} M^{1-r}} \right) D(A; x) \le \frac{1}{4} r \left(\frac{M^{1-r} - m^{1-r}}{m^{1-r} M^{1-r}} \right) (M - m)$$

for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
(iii) If A is invertible, then

$$0 \le \langle A^{-r} x, x \rangle - \langle Ax, x \rangle^{-r} \tag{4.50}$$

$$\le \frac{1}{2} r \left(\frac{M^{r+1} - m^{r+1}}{M^{r+1} m^{r+1}} \right) \langle |\langle Ax, x \rangle \cdot 1_H - A| x, x \rangle$$

$$\le \frac{1}{2} r \left(\frac{M^{r+1} - m^{r+1}}{M^{r+1} m^{r+1}} \right) D(A; x) \le \frac{1}{4} r \left(\frac{M^{r+1} - m^{r+1}}{M^{r+1} m^{r+1}} \right) (M - m)$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

2. Consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$. On utilizing the inequality (4.38), we can state the following result:

Proposition 262 *For any positive definite operator A on the Hilbert space H with $Sp(A) \subseteq [m, M] \subseteq [0, \infty)$ we have the inequality*

$$(0 \le) \ln(\langle Ax, x \rangle) - \langle \ln(A) x, x \rangle \tag{4.51}$$

$$\le \frac{1}{2} \cdot \frac{M - m}{mM} \langle |\langle Ax, x \rangle \cdot 1_H - A| x, x \rangle$$

$$\le \frac{1}{2} \cdot \frac{M - m}{mM} D(A; x) \le \frac{1}{4} \cdot \frac{(M - m)^2}{mM}$$

for any $x \in H$ with $\|x\| = 1$.

Finally, the following result for logarithms also holds:

Proposition 263 *Under the assumptions of Proposition 262 we have the inequality*

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \tag{4.52}$$

$$\leq \ln \sqrt{\frac{M}{m}} \langle |\langle Ax, x \rangle \cdot 1_H - A| x, x \rangle$$

$$\leq \ln \sqrt{\frac{M}{m}} \cdot D(A; x) \leq \frac{1}{2} (M - m) \ln \sqrt{\frac{M}{m}}$$

for any $x \in H$ with $\|x\| = 1$.

Remark 264 *On utilizing the results from the previous sections for other convex functions of interest such as $f(x) = \ln [(1 - x)/x]$, $x \in (0, 1/2)$ or $f(x) = \ln (1 + \exp x)$, $x \in (-\infty, \infty)$ we can get other interesting operator inequalities. However, the details are left to the interested reader.*

4.4 Other Ostrowski Inequalities for Continuous Functions

4.4.1 Inequalities for Absolutely Continuous Functions of Selfadjoint Operators

We start with the following scalar inequality that is of interest in itself since it provides a generalization of the Ostrowski inequality when upper and lower bounds for the derivative are provided:

Lemma 265 (Dragomir, 2010, [27]) *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function whose derivative is bounded above and below on $[a, b]$, i.e., there exists the real constants γ and $\Gamma, \gamma < \Gamma$ with the property that $\gamma \leq f'(s) \leq \Gamma$ for almost every $s \in [a, b]$. Then we have the double inequality*

$$-\frac{1}{2} \cdot \frac{\Gamma - \gamma}{b - a} \left[\left(s - \frac{b\Gamma - a\gamma}{\Gamma - \gamma} \right)^2 - \Gamma \gamma \left(\frac{b - a}{\Gamma - \gamma} \right)^2 \right] \tag{4.53}$$

$$\leq f(s) - \frac{1}{b - a} \int_a^b f(t) \, dt$$

$$\leq \frac{1}{2} \cdot \frac{\Gamma - \gamma}{b - a} \left[\left(s - \frac{a\Gamma - b\gamma}{\Gamma - \gamma} \right)^2 - \Gamma \gamma \left(\frac{b - a}{\Gamma - \gamma} \right)^2 \right]$$

for any $s \in [a, b]$. The inequalities are sharp.

Proof. We start with *Montgomery's identity*

$$f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt \tag{4.54}$$

$$= \frac{1}{b-a} \int_a^s (t-a) f'(t)\, dt + \frac{1}{b-a} \int_s^b (t-b) f'(t)\, dt$$

that holds for any $s \in [a, b]$.

Since $\gamma \le f'(t) \le \Gamma$ for almost every $t \in [a, b]$, then

$$\frac{\gamma}{b-a} \int_a^s (t-a)\, dt \le \frac{1}{b-a} \int_a^s (t-a) f'(t)\, dt \le \frac{\Gamma}{b-a} \int_a^s (t-a)\, dt$$

and

$$\frac{\Gamma}{b-a} \int_s^b (b-t)\, dt \le \frac{1}{b-a} \int_s^b (b-t) f'(t)\, dt \le \frac{\Gamma}{b-a} \int_s^b (b-t)\, dt$$

for any $s \in [a, b]$.

Now, due to the fact that

$$\int_a^s (t-a)\, dt = \frac{1}{2}(s-a)^2 \text{ and } \int_s^b (b-t)\, dt = \frac{1}{2}(b-s)^2$$

then by (4.54) we deduce the following inequality that is of interest in itself:

$$-\frac{1}{2(b-a)} \left[\Gamma(b-s)^2 - \gamma(s-a)^2 \right] \tag{4.55}$$

$$\le f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt$$

$$\le \frac{1}{2(b-a)} \left[\Gamma(s-a)^2 - \gamma(b-s)^2 \right]$$

for any $s \in [a, b]$.

Further on, if we denote by

$$A := \gamma(s-a)^2 - \Gamma(b-s)^2 \text{ and } B := \Gamma(s-a)^2 - \gamma(b-s)^2$$

then, after some elementary calculations, we derive that

$$A = -(\Gamma - \gamma) \left(s - \frac{b\Gamma - a\gamma}{\Gamma - \gamma} \right)^2 + \frac{\Gamma\gamma}{\Gamma - \gamma}(b-a)^2$$

and

$$B = (\Gamma - \gamma) \left(s - \frac{a\Gamma - b\gamma}{\Gamma - \gamma} \right)^2 - \frac{\Gamma\gamma}{\Gamma - \gamma}(b-a)^2$$

which, together with (4.55), produces the desired result (4.53).

The sharpness of the inequalities follow from the sharpness of some particular cases outlined below. The details are omitted. ∎

Corollary 266 *With the assumptions of Lemma 265 we have the inequalities*

$$\frac{1}{2}\gamma\,(b-a) \le \frac{1}{b-a}\int_a^b f(t)\,dt - f(a) \le \frac{1}{2}\Gamma\,(b-a) \qquad (4.56)$$

and

$$\frac{1}{2}\gamma\,(b-a) \le f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{2}\Gamma\,(b-a) \qquad (4.57)$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8}\,(\Gamma-\gamma)\,(b-a) \qquad (4.58)$$

respectively. The constant $\frac{1}{8}$ is best possible in (4.58).

The proof is obvious from (4.53) on choosing $s = a, s = b$ and $s = \frac{a+b}{2}$, respectively.

Corollary 267 (Dragomir, 2010, [27]) *With the assumptions of Lemma 265 and if, in addition $\gamma = -\alpha$ and $\Gamma = \beta$ with $\alpha, \beta > 0$ then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{b\beta+a\alpha}{\beta+\alpha}\right) \le \frac{1}{2}\cdot\alpha\beta\left(\frac{b-a}{\beta+\alpha}\right) \qquad (4.59)$$

and

$$f\left(\frac{a\beta+b\alpha}{\beta+\alpha}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{2}\cdot\alpha\beta\left(\frac{b-a}{\beta+\alpha}\right). \qquad (4.60)$$

The proof follows from (4.53) on choosing $s = \frac{b\beta+a\alpha}{\beta+\alpha} \in [a,b]$ and $s = \frac{a\beta+b\alpha}{\beta+\alpha} \in [a,b]$, respectively.

Remark 268 *If $f : [a,b] \to \mathbb{R}$ is absolutely continuous and*

$$\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)| < \infty,$$

then by choosing $\gamma = -\|f'\|_\infty$ and $\Gamma = \|f'\|_\infty$ in (4.53) we deduce the classical Ostrowski's inequality for absolutely continuous functions. The constant $\frac{1}{4}$ in Ostrowski's inequality is best possible.

We are able now to state the following result providing upper and lower bounds for absolutely convex functions of selfadjoint operators in Hilbert spaces whose derivatives are bounded below and above:

Theorem 269 (Dragomir, 2010, [27]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is an absolutely continuous function such that there exists the real constants γ and $\Gamma, \gamma < \Gamma$ with the property*

that $\gamma \le f'(s) \le \Gamma$ for almost every $s \in [m, M]$, then we have the following double inequality in the operator order of $B(H)$:

$$-\frac{1}{2} \cdot \frac{\Gamma - \gamma}{M - m} \tag{4.61}$$

$$\times \left[\left(A - \frac{M\Gamma - m\gamma}{\Gamma - \gamma} \cdot 1_H \right)^2 - \Gamma\gamma \left(\frac{M - m}{\Gamma - \gamma} \right)^2 \cdot 1_H \right]$$

$$\le f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H$$

$$\le \frac{1}{2} \cdot \frac{\Gamma - \gamma}{M - m}$$

$$\times \left[\left(A - \frac{m\Gamma - M\gamma}{\Gamma - \gamma} \cdot 1_H \right)^2 - \Gamma\gamma \left(\frac{M - m}{\Gamma - \gamma} \right)^2 \cdot 1_H \right].$$

The proof follows by the property (P) applied for the inequality (4.53) in Lemma 265.

Theorem 270 (Dragomir, 2010, [27]) *With the assumptions in Theorem 269 we have in the operator order the following inequalities*

$$\left| f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H \right| \tag{4.62}$$

$$\le \begin{cases} \left[\frac{1}{4} 1_H + \left(\frac{A - \frac{m+M}{2} 1_H}{M - m} \right)^2 \right] (M - m) \|f'\|_\infty, \ \text{if} \ f' \in L_\infty[m, M]; \\[2mm] \frac{1}{(p+1)^{\frac{1}{p}}} \left[\left(\frac{A - m 1_H}{M - m} \right)^{p+1} + \left(\frac{M 1_H - A}{M - m} \right)^{p+1} \right] (M - m)^{\frac{1}{q}} \|f'\|_q \\ \qquad \text{if} \ f' \in L_p[m, M], \ \frac{1}{p} + \frac{1}{q} = 1, \ p > 1; \\[2mm] \left[\frac{1}{2} 1_H + \left| \frac{A - \frac{m+M}{2} 1_H}{M - m} \right| \right] \|f'\|_1. \end{cases}$$

The proof is obvious by the scalar inequalities from Theorem 239 and the property (P).

The third inequality in (4.62) can be naturally generalized for functions of bounded variation as follows:

Theorem 271 (Dragomir, 2010, [27]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is a continuous function of bounded*

variation on $[m, M]$, *then we have the inequality*

$$\left| f\left(A\right) - \left(\frac{1}{M-m} \int_m^M f\left(t\right) dt\right) \cdot 1_H \right| \tag{4.63}$$

$$\leq \left[\frac{1}{2} 1_H + \left| \frac{A - \frac{m+M}{2} 1_H}{M-m} \right| \right] \bigvee_m^M \left(f\right)$$

where $\bigvee_m^M \left(f\right)$ *denotes the total variation of* f *on* $[m, M]$. *The constant* $\frac{1}{2}$ *is best possible in (4.63).*

Proof. Follows from the scalar inequality obtained by the author in [23], namely

$$\left| f\left(s\right) - \frac{1}{b-a} \int_a^b f\left(t\right) dt \right| \leq \left[\frac{1}{2} + \left| \frac{s - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b \left(f\right) \tag{4.64}$$

for any $s \in [a, b]$, *where* f *is a function of bounded variation on* $[a, b]$. *The constant* $\frac{1}{2}$ *is best possible in (4.64).* ∎

4.4.2 Inequalities for Convex Functions of Selfadjoint Operators

The case of convex functions is important for applications.

We need the following lemma.

Lemma 272 (Dragomir, 2010, [27]) *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable convex function such that the derivative* f' *is continuous on* (a, b) *and with the lateral derivative finite and* $f'_-\left(b\right) \neq f'_+\left(a\right)$. *Then we have the following double inequality*

$$-\frac{1}{2} \cdot \frac{f'_-\left(b\right) - f'_+\left(a\right)}{b-a} \tag{4.65}$$

$$\times \left[\left(s - \frac{bf'_-\left(b\right) - af'_+\left(a\right)}{f'_-\left(b\right) - f'_+\left(a\right)}\right)^2 - f'_-\left(b\right) f'_+\left(a\right) \left(\frac{b-a}{f'_-\left(b\right) - f'_+\left(a\right)}\right)^2 \right]$$

$$\leq f\left(s\right) - \frac{1}{b-a} \int_a^b f\left(t\right) dt \leq f'\left(s\right) \left(s - \frac{a+b}{2}\right)$$

for any $s \in [a, b]$.

Proof. Since f is convex, then by the fact that f' is monotonic nondecreasing, we have

$$\frac{f'_+\left(a\right)}{b-a} \int_a^s \left(t - a\right) dt \leq \frac{1}{b-a} \int_a^s \left(t - a\right) f'\left(t\right) dt \leq \frac{f'\left(s\right)}{b-a} \int_a^s \left(t - a\right) dt$$

and

$$\frac{f'(s)}{b-a} \int_s^b (b-t)\, dt \le \frac{1}{b-a} \int_s^b (b-t)\, f'(t)\, dt \le \frac{f'_-(b)}{b-a} \int_s^b (b-t)\, dt$$

for any $s \in [a, b]$, where $f'_+(a)$ and $f'_-(b)$ are the lateral derivatives in a and b respectively.

Utilising the Montgomery identity (4.54) we then have

$$\frac{f'_+(a)}{b-a} \int_a^s (t-a)\, dt - \frac{f'_-(b)}{b-a} \int_s^b (b-t)\, dt$$

$$\le f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt$$

$$\le \frac{f'(s)}{b-a} \int_a^s (t-a)\, dt - \frac{f'(s)}{b-a} \int_s^b (b-t)\, dt$$

which is equivalent with the following inequality that is of interest in itself

$$\frac{1}{2(b-a)} \left[f'_+(a)(s-a)^2 - f'_-(b)(b-s)^2 \right] \qquad (4.66)$$

$$\le f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt \le f'(s) \left(s - \frac{a+b}{2} \right)$$

for any $s \in [a, b]$.

A simple calculation reveals now that the left side of (4.66) coincides with the same side of the desired inequality (4.65). ∎

We are able now to sate our result for convex functions of selfadjoint operators:

Theorem 273 (Dragomir, 2010, [27]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is a differentiable convex function such that the derivative f' is continuous on (m, M) and with the lateral derivative finite and $f'_-(M) \ne f'_+(m)$, then we have the double inequality in the operator order of $B(H)$*

$$-\frac{1}{2} \cdot \frac{f'_-(M) - f'_+(m)}{M - m} \qquad (4.67)$$

$$\times \left[\left(A - \frac{M f'_-(M) - m f'_+(m)}{f'_-(M) - f'_+(m)} \cdot 1_H \right)^2 \right.$$

$$\left. - f'_-(M) f'_+(m) \left(\frac{M - m}{f'_-(M) - f'_+(m)} \right)^2 \cdot 1_H \right]$$

$$\le f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H \le \left(A - \frac{m + M}{2} \cdot 1_H \right) f'(A).$$

The proof follows from the scalar case in Lemma 272.

Remark 274 *We observe that one can drop the assumption of differentiability of the convex function and will still have the first inequality in (4.67). This follows from the fact that the class of differentiable convex functions is dense in the class of all convex functions defined on a given interval.*

A different lower bound for the quantity

$$f(A) - \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) \cdot 1_H$$

expressed only in terms of the operator A and not its second power as above, also holds:

Theorem 275 (Dragomir, 2010, [27]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is a convex function on $[m, M]$, then we have the following inequality in the operator order of $B(H)$*

$$f(A) - \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) \cdot 1_H \qquad (4.68)$$

$$\geq \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) \cdot 1_H$$

$$- \frac{f(M)(M \cdot 1_H - A) + f(m)(A - m \cdot 1_H)}{M-m}.$$

Proof. It suffices to prove for the case of differentiable convex functions defined on (m, M).

So, by the gradient inequality we have that

$$f(t) - f(s) \geq (t - s) f'(s)$$

for any $t, s \in (m, M)$.

Now, if we integrate this inequality over $s \in [m, M]$ we get

$$(M-m) f(t) - \int_m^M f(s)\, ds \qquad (4.69)$$

$$\geq \int_m^M (t-s) f'(s)\, ds$$

$$= \int_m^M f(s)\, ds - (M-t) f(M) - (t-m) f(m)$$

for each $s \in [m, M]$.

Finally, if we apply to the inequality (4.69) the property (P), we deduce the desired result (4.68). ■

Corollary 276 (Dragomir, 2010, [27]) *With the assumptions of Theorem 275 we have the following double inequality in the operator order*

$$\frac{f\left(m\right)+f\left(M\right)}{2}\cdot 1_{H} \tag{4.70}$$

$$\geq \frac{1}{2}\left[f\left(A\right)+\frac{f\left(M\right)\left(M\cdot 1_{H}-A\right)+f\left(m\right)\left(A-m\cdot 1_{H}\right)}{M-m}\right]$$

$$\geq \left(\frac{1}{M-m}\int_{m}^{M}f\left(t\right)dt\right)\cdot 1_{H}.$$

Proof. The second inequality is equivalent with (4.68).

For the first inequality, we observe, by the convexity of f we have that

$$\frac{f\left(M\right)\left(t-m\right)+f\left(m\right)\left(M-t\right)}{M-m}\geq f\left(t\right)$$

for any $t\in\left[m,M\right]$, which produces the operator inequality

$$\frac{f\left(M\right)\left(A-m\cdot 1_{H}\right)+f\left(m\right)\left(M\cdot 1_{H}-A\right)}{M-m}\geq f\left(A\right). \tag{4.71}$$

Now, if in both sides of (4.71) we add the same quantity

$$\frac{f\left(M\right)\left(M\cdot 1_{H}-A\right)+f\left(m\right)\left(A-m\cdot 1_{H}\right)}{M-m}$$

and perform the calculations, then we obtain the first part of (4.70) and the proof is complete. ∎

4.4.3 Some Vector Inequalities

The following result holds:

Theorem 277 (Dragomir, 2010, [27]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\left(A\right)\subseteq\left[m,M\right]$ for some real numbers $m<M$ and let $\left\{E_{\lambda}\right\}_{\lambda}$ be its spectral family. If $f:\left[m,M\right]\to\mathbb{R}$ is*

an absolutely continuous function on $[m, M]$, *then we have the inequalities*

$$|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \tag{4.72}$$

$$\leq \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right)$$

$$\times \begin{cases} \left[\frac{1}{2}(M - m) + \left|s - \frac{m+M}{2}\right|\right] \|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ \left[\frac{1}{2}(M - m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q} \|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

$$\leq \|x\| \|y\|$$

$$\times \begin{cases} \left[\frac{1}{2}(M - m) + \left|s - \frac{m+M}{2}\right|\right] \|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ \left[\frac{1}{2}(M - m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q} \|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

for any $x, y \in H$ *and* $s \in [m, M]$.

Proof. Since f is absolutely continuous, then we have

$$|f(s) - f(t)| \tag{4.73}$$

$$= \left|\int_s^t f'(u)\, du\right| \leq \left|\int_s^t |f'(u)|\, du\right|$$

$$\leq \begin{cases} |t - s| \|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ |t - s|^{1/q} \|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

for any $s, t \in [m, M]$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous functions and $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\, dv(t)\right| \leq \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v),$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Now, by the above property of the Riemann-Stieltjes integral we have from the representation (4.79) that

$$|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \tag{4.74}$$

$$= \left| \int_{m-0}^{M} [f(s) - f(t)] \, d(\langle E_t x, y\rangle) \right|$$

$$\leq \max_{t \in [m,M]} |f(s) - f(t)| \bigvee_{m}^{M} (\langle E_{(\cdot)}x, y\rangle)$$

$$\leq \bigvee_{m}^{M} (\langle E_{(\cdot)}x, y\rangle)$$

$$\times \begin{cases} \max_{t \in [m,M]} |t-s| \, \|f'\|_\infty & \text{if } f' \in L_\infty[m,M] \\[2mm] \max_{t \in [m,M]} |t-s|^{1/q} \, \|f'\|_p & \begin{array}{l} \text{if } f' \in L_p[m,M], p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \end{cases}$$

$$:= F$$

where $\bigvee_{m}^{M} (\langle E_{(\cdot)}x, y\rangle)$ denotes the total variation of $\langle E_{(\cdot)}x, y\rangle$ and $x, y \in H$.

Since, obviously, we have $\max_{t \in [m,M]} |t-s| = \frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right|$, then

$$F = \bigvee_{m}^{M} (\langle E_{(\cdot)}x, y\rangle) \tag{4.75}$$

$$\times \begin{cases} \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right] \|f'\|_\infty & \text{if } f' \in L_\infty[m,M] \\[2mm] \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q} \|f'\|_p & \begin{array}{l} \text{if } f' \in L_p[m,M], p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \end{cases}$$

for any $x, y \in H$.

The last part follows by the Total Variation Schwarz's inequality and the details are omitted. ∎

Corollary 278 (Dragomir, 2010, [27]) *With the assumptions of Theorem 277 we have the following inequalities*

$$\left| f\left(\frac{\langle Ax, x\rangle}{\|x\|^2} \right) \langle x, y\rangle - \langle f(A)x, y\rangle \right| \tag{4.76}$$

$$\leq \|x\|\,\|y\|$$

$$\times \begin{cases} \left[\frac{1}{2}(M-m) + \left| \frac{\langle Ax, x\rangle}{\|x\|^2} - \frac{m+M}{2} \right| \right] \|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\[2ex] \left[\frac{1}{2}(M-m) + \left| \frac{\langle Ax, x\rangle}{\|x\|^2} - \frac{m+M}{2} \right| \right]^{1/q} \|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

and

$$\left| f\left(\frac{m+M}{2} \right) \langle x, y\rangle - \langle f(A)x, y\rangle \right| \tag{4.77}$$

$$\leq \|x\|\,\|y\|$$

$$\times \begin{cases} \frac{1}{2}(M-m)\|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\[2ex] \frac{1}{2^{1/q}}(M-m)^{1/q}\|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

for any $x, y \in H$.

Remark 279 *In particular, we obtain from (4.60) the following inequalities*

$$|f(\langle Ax, x\rangle) - \langle f(A)x, x\rangle| \tag{4.78}$$

$$\leq \begin{cases} \left[\frac{1}{2}(M-m) + \left| \langle Ax, x\rangle - \frac{m+M}{2} \right| \right] \|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\[2ex] \left[\frac{1}{2}(M-m) + \left| \langle Ax, x\rangle - \frac{m+M}{2} \right| \right]^{1/q} \|f'\|_p & \text{if } f' \in L_p[m, M], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

and

$$\left| f\left(\frac{m+M}{2} \right) - \langle f(A)x, x\rangle \right| \tag{4.79}$$

$$\leq \begin{cases} \frac{1}{2}(M-m)\|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\[2ex] \frac{1}{2^{1/q}}(M-m)^{1/q}\|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Theorem 280 (Dragomir, 2010, [27]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real*

numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is
$r - H$-Hölder continuous on $[m, M]$, then we have the inequality

$$|f(s)\langle x, y\rangle - \langle f(A) x, y\rangle| \tag{4.80}$$

$$\leq H \bigvee_m^M \left(\langle E_{(\cdot)} x, y\rangle\right) \left[\frac{1}{2}(M - m) + \left|s - \frac{m + M}{2}\right|\right]^r$$

$$\leq H \|x\| \|y\| \left[\frac{1}{2}(M - m) + \left|s - \frac{m + M}{2}\right|\right]^r$$

for any $x, y \in H$ and $s \in [m, M]$.
  In particular, we have the inequalities

$$\left|f\left(\frac{\langle Ax, x\rangle}{\|x\|^2}\right)\langle x, y\rangle - \langle f(A) x, y\rangle\right| \tag{4.81}$$

$$\leq H \|x\| \|y\| \left[\frac{1}{2}(M - m) + \left|\frac{\langle Ax, x\rangle}{\|x\|^2} - \frac{m + M}{2}\right|\right]^r$$

and

$$\left|f\left(\frac{m + M}{2}\right)\langle x, y\rangle - \langle f(A) x, y\rangle\right| \leq \frac{1}{2^r} H \|x\| \|y\| (M - m)^r \tag{4.82}$$

for any $x, y \in H$.

Proof. Utilising the inequality (4.74) and the fact that f is $r - H$-Hölder
continuous we have successively

$$|f(s)\langle x, y\rangle - \langle f(A) x, y\rangle| \tag{4.83}$$

$$= \left|\int_{m-0}^M [f(s) - f(t)] \, d(\langle E_t x, y\rangle)\right|$$

$$\leq \max_{t \in [m, M]} |f(s) - f(t)| \bigvee_m^M \left(\langle E_{(\cdot)} x, y\rangle\right)$$

$$\leq H \max_{t \in [m, M]} |s - t|^r \bigvee_m^M \left(\langle E_{(\cdot)} x, y\rangle\right)$$

$$= H \left[\frac{1}{2}(M - m) + \left|s - \frac{m + M}{2}\right|\right]^r \bigvee_m^M \left(\langle E_{(\cdot)} x, y\rangle\right)$$

for any $x, y \in H$ and $s \in [m, M]$.
  The argument follows now as in the proof of Theorem 277 and the details
are omitted. ∎

4.4.4   Logarithmic Inequalities

Consider *the identric mean*

$$I = I\left(a, b\right) := \begin{cases} a & \text{if} \quad a = b, \\[2mm] \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b, \end{cases} \qquad a, b > 0;$$

and observe that

$$\frac{1}{b-a}\int_a^b \ln t\, dt = \ln\left[I\left(a, b\right)\right].$$

If we apply Theorem 273 for the convex function $f\left(t\right) = -\ln t, t > 0$, then we can state:

Proposition 281 *Let A be a positive selfadjoint operator in the Hilbert space H with the spectrum $Sp\left(A\right) \subseteq [m, M]$ for some positive numbers $0 < m < M$. Then we have the double inequality in the operator order of $B\left(H\right)$*

$$-\frac{1}{2mM}\left(A^2 - mM\right) \leq \ln I\left(m, M\right)\cdot 1_H - \ln A \leq \frac{m + M}{2}\cdot A^{-1} - 1_H. \quad (4.84)$$

If we denote by $G\left(a, b\right) := \sqrt{ab}$ the geometric mean of the positive numbers a, b, then we can state the following result as well:

Proposition 282 *With the assumptions of Proposition 281, we have the inequalities in the operator order of $B\left(H\right)$*

$$\ln G\left(m, M\right)\cdot 1_H \qquad\qquad\qquad\qquad (4.85)$$
$$\leq \frac{1}{2}\left[\ln A + \frac{\ln M\cdot\left(M\cdot 1_H - A\right) + \ln m\cdot\left(A - m\cdot 1_H\right)}{M - m}\right]$$
$$\leq \ln I\left(m, M\right)\cdot 1_H.$$

The inequality follows by Corollary 276 applied for the convex function $f\left(t\right) = -\ln t, t > 0$.

Finally, the following vector inequality may be stated

Proposition 283 *With the assumptions of Proposition 281, for any $x, y \in H$ we have the inequalities*

$$\left|\langle x, y\rangle \ln s - \langle \ln Ax, y\rangle\right| \qquad\qquad\qquad\qquad (4.86)$$
$$\leq \|x\|\,\|y\| \begin{cases} \left[\frac{1}{2}\left(M - m\right) + \left|s - \frac{m+M}{2}\right|\right]\frac{1}{m}, \\[3mm] \left[\frac{1}{2}\left(M - m\right) + \left|s - \frac{m+M}{2}\right|\right]^{1/q}\frac{M^{p-1} - m^{p-1}}{(p-1)M^{p-1}m^{p-1}}, \end{cases}$$

for any $s \in [m, M]$, where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

4.5   More Ostrowski's Type Inequalities

4.5.1   Some Vector Inequalities for Functions of Bounded Variation

The following result holds:

Theorem 284 (Dragomir, 2010, [16]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$|f(s) \langle x, y \rangle - \langle f(A) x, y \rangle| \tag{4.87}$$

$$\leq \langle E_s x, x \rangle^{1/2} \langle E_s y, y \rangle^{1/2} \bigvee_m^s (f)$$

$$+ \langle (1_H - E_s) x, x \rangle^{1/2} \langle (1_H - E_s) y, y \rangle^{1/2} \bigvee_s^M (f)$$

$$\leq \|x\| \|y\| \left(\frac{1}{2} \bigvee_m^M (f) + \frac{1}{2} \left| \bigvee_m^s (f) - \bigvee_s^M (f) \right| \right) \left(\leq \|x\| \|y\| \bigvee_m^M (f) \right)$$

for any $x, y \in H$ and for any $s \in [m, M]$.

Proof. We use the following identity for the Riemann-Stieltjes integral established by the author in 2000 in [10] (see also [31, p. 452]):

$$[u(b) - u(a)] f(s) - \int_a^b f(t) \, du(t) \tag{4.88}$$

$$= \int_a^s [u(t) - u(a)] \, df(t) + \int_s^b [u(t) - u(b)] \, df(t),$$

for any $s \in [a, b]$, provided the Riemann-Stieltjes integral $\int_a^b f(t) \, du(t)$ exists.

A simple proof can be done by utilizing the integration by parts formula and starting from the right hand side of (4.88).

If we choose in (4.88) $a = m, b = M$ and $u(t) = \langle E_t x, y \rangle$, then we have the following identity of interest in itself

$$f(s) \langle x, y \rangle - \langle f(A) x, y \rangle \tag{4.89}$$

$$= \int_{m-0}^s \langle E_t x, y \rangle \, df(t) + \int_s^M \langle (E_t - 1_H) x, y \rangle \, df(t)$$

for any $x, y \in H$ and for any $s \in [m, M]$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function and $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v)$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising this property we have from (4.89) that

$$|f(s) \langle x, y \rangle - \langle f(A) x, y \rangle| \tag{4.90}$$

$$\leq \left| \int_{m-0}^{s} \langle E_t x, y \rangle \, df(t) \right| + \left| \int_{s}^{M} \langle (E_t - 1_H) x, y \rangle \, df(t) \right|$$

$$\leq \max_{t \in [m,s]} |\langle E_t x, y \rangle| \bigvee_m^s (f) + \max_{t \in [s,M]} |\langle (E_t - 1_H) x, y \rangle| \bigvee_s^M (f) := T$$

for any $x, y \in H$ and for any $s \in [m, M]$.

If P is a nonnegative operator on H, i.e., $\langle P x, x \rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle P x, y \rangle|^2 \leq \langle P x, x \rangle \langle P y, y \rangle \tag{4.91}$$

for any $x, y \in H$.

On applying the inequality (4.91) we have

$$|\langle E_t x, y \rangle| \leq \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2}$$

and

$$|\langle (1_H - E_t) x, y \rangle| \leq \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2}$$

for any $x, y \in H$ and $t \in [m, M]$.

Therefore

$$T \le \max_{t\in[m,s]} \left[\langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \right] \bigvee_m^s (f) \tag{4.92}$$

$$+ \max_{t\in[s,M]} \left[\langle (1_H - E_t)\, x, x\rangle^{1/2} \langle (1_H - E_t)\, y, y\rangle^{1/2} \right] \bigvee_s^M (f)$$

$$\le \max_{t\in[m,s]} \langle E_t x, x\rangle^{1/2} \max_{t\in[m,s]} \langle E_t y, y\rangle^{1/2} \bigvee_m^s (f)$$

$$+ \max_{t\in[s,M]} \langle (1_H - E_t)\, x, x\rangle^{1/2} \max_{t\in[s,M]} \langle (1_H - E_t)\, y, y\rangle^{1/2} \bigvee_s^M (f)$$

$$= \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f)$$

$$+ \langle (1_H - E_s)\, x, x\rangle^{1/2} \langle (1_H - E_s)\, y, y\rangle^{1/2} \bigvee_s^M (f)$$

$$:= V$$

for any $x, y \in H$ and for any $s \in [m, M]$, proving the first inequality in (4.87).

Now, observe that

$$V \le \max \left\{ \bigvee_m^s (f), \bigvee_s^M (f) \right\}$$

$$\times \left[\langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} + \langle (1_H - E_s)\, x, x\rangle^{1/2} \langle (1_H - E_s)\, y, y\rangle^{1/2} \right].$$

Since

$$\max \left\{ \bigvee_m^s (f), \bigvee_s^M (f) \right\} = \frac{1}{2} \bigvee_m^M (f) + \frac{1}{2} \left| \bigvee_m^s (f) - \bigvee_s^M (f) \right|$$

and by the Cauchy-Buniakovski-Schwarz inequality for positive real numbers a_1, b_1, a_2, b_2

$$a_1 b_1 + a_2 b_2 \le \left(a_1^2 + a_2^2 \right)^{1/2} \left(b_1^2 + b_2^2 \right)^{1/2} \tag{4.93}$$

we have

$$\langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} + \langle (1_H - E_s)\, x, x\rangle^{1/2} \langle (1_H - E_s)\, y, y\rangle^{1/2}$$

$$\le \left[\langle E_s x, x\rangle + \langle (1_H - E_s)\, x, x\rangle \right]^{1/2} \left[\langle E_s y, y\rangle + \langle (1_H - E_s)\, y, y\rangle \right]^{1/2}$$

$$= \|x\| \, \|y\|$$

for any $x, y \in H$ and $s \in [m, M]$, then the last part of (4.87) is proven as well. ∎

Remark 285 *For the continuous function with bounded variation $f : [m, M] \to$ $\mathbb{R}$ if $p \in [m, M]$ is a point with the property that*

$$\bigvee_{m}^{p} (f) = \bigvee_{p}^{M} (f)$$

then from (4.87) we get the interesting inequality

$$|f(p) \langle x, y \rangle - \langle f(A) x, y \rangle| \le \frac{1}{2} \|x\| \|y\| \bigvee_{m}^{M} (f) \qquad (4.94)$$

for any $x, y \in H$.

   If the continuous function $f : [m, M] \to \mathbb{R}$ is monotonic nondecreasing and therefore of bounded variation, we get from (4.87) the following inequality as well

$$|f(s) \langle x, y \rangle - \langle f(A) x, y \rangle| \qquad (4.95)$$
$$\le \langle E_s x, x \rangle^{1/2} \langle E_s y, y \rangle^{1/2} (f(s) - f(m))$$
$$+ \langle (1_H - E_s) x, x \rangle^{1/2} \langle (1_H - E_s) y, y \rangle^{1/2} (f(M) - f(s))$$
$$\le \|x\| \|y\| \left(\frac{1}{2} (f(M) - f(m)) + \left| f(s) - \frac{f(m) + f(M)}{2} \right| \right)$$
$$(\le \|x\| \|y\| f(M) - f(m))$$

for any $x, y \in H$ and $s \in [m, M]$.

   Moreover, if the continuous function $f : [m, M] \to \mathbb{R}$ is nondecreasing on $[m, M]$, then the equation

$$f(s) = \frac{f(m) + f(M)}{2}$$

has got at least a solution in $[m, M]$. In his case we get from (4.95) the following trapezoidal type inequality

$$\left| \frac{f(m) + f(M)}{2} \langle x, y \rangle - \langle f(A) x, y \rangle \right| \qquad (4.96)$$
$$\le \frac{1}{2} \|x\| \|y\| (f(M) - f(m))$$

for any $x, y \in H$.

4.5.2   Some Vector Inequalities for Lipshitzian Functions

The following result that incorporates the case of Lipschitzian functions also holds

Theorem 286 (Dragomir, 2010, [16]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, i.e.,*

$$|f(s) - f(t)| \leq L|s - t| \text{ for any } s, t \in [m, M],$$

then we have the inequality

$$|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \tag{4.97}$$

$$\leq L\left[\left(\int_{m-0}^{s}\langle E_t x, x\rangle \, dt\right)^{1/2}\left(\int_{m-0}^{s}\langle E_t y, y\rangle \, dt\right)^{1/2}\right.$$

$$\left. + \left(\int_{s}^{M}\langle(1_H - E_t)x, x\rangle \, dt\right)^{1/2}\left(\int_{s}^{M}\langle(1_H - E_t)y, y\rangle \, dt\right)^{1/2}\right]$$

$$\leq L\langle|A - s1_H|x, x\rangle^{1/2}\langle|A - s1_H|y, y\rangle^{1/2}$$

$$\leq L\left[D^2(A; x) + \left(s\|x\|^2 - \langle Ax, x\rangle\right)^2\right]^{1/4}$$

$$\times \left[D^2(A; y) + \left(s\|y\|^2 - \langle Ay, y\rangle\right)^2\right]^{1/4}$$

for any $x, y \in H$ and $s \in [m, M]$, where $D(A; x)$ is the variance of the selfadjoint operator A in x and is defined by

$$D(A; x) := \left(\|Ax\|^2 \|x\|^2 - \langle Ax, x\rangle^2\right)^{1/2}.$$

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t) \, dv(t)\right| \leq L\int_a^b |p(t)| \, dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (4.89) that

$$|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle|$$

$$\leq \left|\int_{m-0}^{s}\langle E_t x, y\rangle \, df(t)\right| + \left|\int_{s}^{M}\langle(E_t - 1_H)x, y\rangle \, df(t)\right|$$

$$\leq L\left[\int_{m-0}^{s}|\langle E_t x, y\rangle| \, dt + \int_{s}^{M}|\langle(E_t - 1_H)x, y\rangle| \, dt\right] := LW$$

for any $x, y \in H$ and $s \in [m, M]$.

By utilizing the generalized Schwarz inequality for nonnegative operators (4.91) and the Cauchy-Buniakovski-Schwarz inequality for the Riemann integral we have

$$
\begin{aligned}
W &\leq \int_{m-0}^{s} \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \, dt \\
&\quad + \int_{s}^{M} \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \, dt \\
&\leq \left(\int_{m-0}^{s} \langle E_t x, x \rangle \, dt \right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y \rangle \, dt \right)^{1/2} \\
&\quad + \left(\int_{s}^{M} \langle (1_H - E_t) x, x \rangle \, dt \right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t) y, y \rangle \, dt \right)^{1/2} \\
&:= Z
\end{aligned}
\tag{4.98}
$$

for any $x, y \in H$ and $s \in [m, M]$.

On the other hand, by making use of the elementary inequality (4.93) we also have

$$
\begin{aligned}
Z &\leq \left(\int_{m-0}^{s} \langle E_t x, x \rangle \, dt + \int_{s}^{M} \langle (1_H - E_t) x, x \rangle \, dt \right)^{1/2} \\
&\quad \times \left(\int_{m-0}^{s} \langle E_t y, y \rangle \, dt + \int_{s}^{M} \langle (1_H - E_t) y, y \rangle \, dt \right)^{1/2}
\end{aligned}
\tag{4.99}
$$

for any $x, y \in H$ and $s \in [m, M]$.

Now, observe that, by the use of the representation (4.89) for the continuous function $f : [m, M] \to \mathbb{R}$, $f(t) = |t - s|$ where s is fixed in $[m, M]$ we have the following identity that is of interest in itself

$$
\langle |A - s \cdot 1_H| \, x, y \rangle = \int_{m-0}^{s} \langle E_t x, y \rangle \, dt + \int_{s}^{M} \langle (1_H - E_t) x, y \rangle \, dt \tag{4.100}
$$

for any $x, y \in H$.

On utilizing (4.100) for x and then for y we deduce the second part of (4.97).

Finally, by the well known inequality for the modulus of a bounded linear operator

$$
\langle |T| \, x, x \rangle \leq \|Tx\| \, \|x\| \, , x \in H
$$

we have

$$\langle |A - s \cdot 1_H| x, x \rangle^{1/2} \leq \|Ax - sx\|^{1/2} \|x\|^{1/2}$$

$$= \left(\|Ax\|^2 - 2s \langle Ax, x \rangle + s^2 \|x\|^2 \right)^{1/4} \|x\|^{1/2}$$

$$= \left[\|Ax\|^2 \|x\|^2 - \langle Ax, x \rangle^2 + \left(s \|x\|^2 - \langle Ax, x \rangle \right)^2 \right]^{1/4}$$

$$= \left[D^2 (A; x) + \left(s \|x\|^2 - \langle Ax, x \rangle \right)^2 \right]^{1/4}$$

and a similar relation for y. The proof is thus complete. ∎

Remark 287 *Since A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, then*

$$\left| A - \frac{m+M}{2} \cdot 1_H \right| \leq \frac{M - m}{2} 1_H$$

giving from (4.97) that

$$\left| f \left(\frac{m+M}{2} \right) \langle x, y \rangle - \langle f(A) x, y \rangle \right| \tag{4.101}$$

$$\leq L \left[\left(\int_{m-0}^{\frac{m+M}{2}} \langle E_t x, x \rangle \, dt \right)^{1/2} \left(\int_{m-0}^{\frac{m+M}{2}} \langle E_t y, y \rangle \, dt \right)^{1/2} \right.$$

$$\left. + \left(\int_{\frac{m+M}{2}}^{M} \langle (1_H - E_t) x, x \rangle \, dt \right)^{1/2} \left(\int_{\frac{m+M}{2}}^{M} \langle (1_H - E_t) y, y \rangle \, dt \right)^{1/2} \right]$$

$$\leq L \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle^{1/2} \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle^{1/2}$$

$$\leq \frac{1}{2} L (M - m) \|x\| \|y\|$$

for any $x, y \in H$.

The particular case of equal vectors is of interest:

Corollary 288 (Dragomir, 2010, [16]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$\left| f(s) \|x\|^2 - \langle f(A) x, x \rangle \right| \tag{4.102}$$

$$\leq L \langle |A - s \cdot 1_H| x, x \rangle$$

$$\leq L \left[D^2 (A; x) + \left(s \|x\|^2 - \langle Ax, x \rangle \right)^2 \right]^{1/2}$$

for any $x \in H$ and $s \in [m, M]$.

Remark 289 *An important particular case that can be obtained from (4.102) is the one when $s = \frac{\langle Ax, x \rangle}{\|x\|^2}, x \neq 0$, giving the inequality*

$$\left| f\left(\frac{\langle Ax, x \rangle}{\|x\|^2} \right) \|x\|^2 - \langle f(A) x, x \rangle \right| \tag{4.103}$$

$$\leq L \left\langle \left| A - \frac{\langle Ax, x \rangle}{\|x\|^2} \cdot 1_H \right| x, x \right\rangle$$

$$\leq LD(A; x) \leq \frac{1}{2} L(M - m) \|x\|^2$$

for any $x \in H, x \neq 0$.

We are able now to provide the following corollary:

Corollary 290 (Dragomir, 2010, [16]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a $(\varphi, \Phi) - Lipschitzian$ functions on $[m, M]$ with $\Phi > \varphi$, then we have the inequality*

$$\left| \langle f(A) x, y \rangle - \frac{\Phi + \varphi}{2} \langle Ax, y \rangle + \frac{\Phi + \varphi}{2} s \langle x, y \rangle - f(s) \langle x, y \rangle \right| \tag{4.104}$$

$$\leq \frac{1}{2} (\Phi - \varphi) \left[\left(\int_{m-0}^s \langle E_t x, x \rangle \, dt \right)^{1/2} \left(\int_{m-0}^s \langle E_t y, y \rangle \, dt \right)^{1/2} \right.$$

$$\left. + \left(\int_s^M \langle (1_H - E_t) x, x \rangle \, dt \right)^{1/2} \left(\int_s^M \langle (1_H - E_t) y, y \rangle \, dt \right)^{1/2} \right]$$

$$\leq \frac{1}{2} (\Phi - \varphi) \langle |A - s1_H| x, x \rangle^{1/2} \langle |A - s1_H| y, y \rangle^{1/2}$$

$$\leq \frac{1}{2} (\Phi - \varphi) \left[D^2(A; x) + \left(s \|x\|^2 - \langle Ax, x \rangle \right)^2 \right]^{1/4}$$

$$\times \left[D^2(A; y) + \left(s \|y\|^2 - \langle Ay, y \rangle \right)^2 \right]^{1/4}$$

for any $x, y \in H$.

Remark 291 *Various particular cases can be stated by utilizing the inequality (4.104), however the details are left to the interested reader.*

4.6 Some Vector Inequalities for Monotonic Functions

The case of monotonic functions is of interest as well. The corresponding result is incorporated in the following

Theorem 292 (Dragomir, 2010, [16]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on $[m, M]$, then we have the inequality*

$$|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \tag{4.105}$$

$$\leq \left(\int_{m-0}^{s} \langle E_t x, x\rangle \, df(t)\right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y\rangle \, df(t)\right)^{1/2}$$

$$+ \left(\int_{s}^{M} \langle (1_H - E_t)x, x\rangle \, df(t)\right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t)y, y\rangle \, df(t)\right)^{1/2}$$

$$\leq \langle |f(A) - f(s)1_H| x, x\rangle^{1/2} \langle |f(A) - f(s)1_H| y, y\rangle^{1/2}$$

$$\leq \left[D^2(f(A); x) + \left(f(s)\|x\|^2 - \langle f(A)x, x\rangle\right)^2\right]^{1/4}$$

$$\times \left[D^2(f(A); y) + \left(f(s)\|y\|^2 - \langle f(A)y, y\rangle\right)^2\right]^{1/4}$$

for any $x, y \in H$ and $s \in [m, M]$, where, as above $D(f(A); x)$ is the variance of the selfadjoint operator $f(A)$ in x.

Proof. From the theory of Riemann-Stieltjes integral is well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t) \, dv(t)$ and $\int_a^b |p(t)| \, dv(t)$ exist and

$$\left|\int_a^b p(t) \, dv(t)\right| \leq \int_a^b |p(t)| \, dv(t).$$

On utilizing this property and the representation (4.89) we have successively

$$|f(s)\langle x, y\rangle - \langle f(A) x, y\rangle| \tag{4.106}$$

$$\leq \left|\int_{m-0}^{s} \langle E_t x, y\rangle \, df(t)\right| + \left|\int_{s}^{M} \langle (E_t - 1_H) x, y\rangle \, df(t)\right|$$

$$\leq \int_{m-0}^{s} |\langle E_t x, y\rangle| \, df(t) + \int_{s}^{M} |\langle (E_t - 1_H) x, y\rangle| \, df(t)$$

$$\leq \int_{m-0}^{s} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \, df(t)$$

$$+ \int_{s}^{M} \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \, df(t)$$

$$:= Y,$$

for any $x, y \in H$ and $s \in [m, M]$.

We use now the following version of the Cauchy-Buniakovski-Schwarz inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators

$$\left(\int_{a}^{b} p(t) q(t) \, dv(t)\right)^2 \leq \int_{a}^{b} p^2(t) \, dv(t) \int_{a}^{b} q^2(t) \, dv(t)$$

to get that

$$\int_{m-0}^{s} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \, df(t)$$

$$\leq \left(\int_{m-0}^{s} \langle E_t x, x\rangle \, df(t)\right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y\rangle \, df(t)\right)^{1/2}$$

and

$$\int_{s}^{M} \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \, df(t)$$

$$\leq \left(\int_{s}^{M} \langle (1_H - E_t) x, x\rangle \, df(t)\right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t) y, y\rangle \, df(t)\right)^{1/2}$$

for any $x, y \in H$ and $s \in [m, M]$.

Therefore

$$Y \leq \left(\int_{m-0}^{s} \langle E_t x, x \rangle \, df(t) \right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y \rangle \, df(t) \right)^{1/2}$$

$$+ \left(\int_{s}^{M} \langle (1_H - E_t) x, x \rangle \, df(t) \right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t) y, y \rangle \, df(t) \right)^{1/2}$$

$$\leq \left(\int_{m-0}^{s} \langle E_t x, x \rangle \, df(t) + \int_{s}^{M} \langle (1_H - E_t) x, x \rangle \, df(t) \right)^{1/2}$$

$$\times \left(\int_{m-0}^{s} \langle E_t y, y \rangle \, df(t) + \int_{s}^{M} \langle (1_H - E_t) y, y \rangle \, df(t) \right)^{1/2}$$

for any $x, y \in H$ and $s \in [m, M]$, where, to get the last inequality we have used the elementary inequality (4.93).

Now, since f is monotonic nondecreasing, on applying the representation (4.89) for the function $|f(\cdot) - f(s)|$ with s fixed in $[m, M]$ we deduce the following identity that is of interest in itself as well:

$$\langle |f(A) - f(s)| x, y \rangle \tag{4.107}$$

$$= \int_{m-0}^{s} \langle E_t x, y \rangle \, df(t) + \int_{s}^{M} \langle (1_H - E_t) x, y \rangle \, df(t)$$

for any $x, y \in H$.

The second part of (4.105) follows then by writing (4.107) for x then by y and utilizing the relevant inequalities from above.

The last part is similar to the corresponding one from the proof of Theorem 286 and the details are omitted. ∎

The following corollary is of interest:

Corollary 293 (Dragomir, 2010, [16]) *With the assumption of Theorem 292 we have the inequalities*

$$\left| \frac{f(m) + f(M)}{2} \langle x, y \rangle - \langle f(A) x, y \rangle \right| \tag{4.108}$$

$$\leq \left\langle \left| f(A) - \frac{f(m) + f(M)}{2} \cdot 1_H \right| x, x \right\rangle^{1/2}$$

$$\times \left\langle \left| f(A) - \frac{f(m) + f(M)}{2} \cdot 1_H \right| y, y \right\rangle^{1/2}$$

$$\leq \frac{1}{2} (f(M) - f(m)) \|x\| \|y\|,$$

for any $x, y \in H$.

Proof. Since f is monotonic nondecreasing, then $f(u) \in [f(m), f(M)]$ for any $u \in [m, M]$. By the continuity of f it follows that there exists at list one $s \in [m, M]$ such that

$$f(s) = \frac{f(m) + f(M)}{2}.$$

Now, on utilizing the inequality (4.105) for this s we deduce the first inequality in (4.108). The second part follows as above and the details are omitted. ∎

4.6.1 Power Inequalities

We consider the power function $f(t) := t^p$ where $p \in \mathbb{R} \setminus \{0\}$ and $t > 0$. The following mid-point inequalities hold:

Proposition 294 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 \le m < M$.*
If $p > 0$, then for any $x, y \in H$

$$\left| \left(\frac{m+M}{2} \right)^p \langle x, y \rangle - \langle A^p x, y \rangle \right| \tag{4.109}$$

$$\le B_p \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle^{1/2} \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle^{1/2}$$

$$\le \frac{1}{2} B_p (M - m) \|x\| \|y\|$$

where

$$B_p = p \times \begin{cases} M^{p-1} & \text{if } p \ge 1 \\ \\ m^{p-1} & \text{if } 0 < p < 1, m > 0. \end{cases}$$

and

$$\left| \left(\frac{m+M}{2} \right)^{-p} \langle x, y \rangle - \langle A^{-p} x, y \rangle \right| \tag{4.110}$$

$$\le C_p \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle^{1/2} \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle^{1/2}$$

$$\le \frac{1}{2} C_p (M - m) \|x\| \|y\|$$

where

$$C_p = p m^{-p-1} \text{ and } m > 0.$$

The proof follows from (4.101).

We can also state the following trapezoidal type inequalities:

Proposition 295 *With the assumption of Proposition 294 and if $p > 0$ we have the inequalities*

$$\left| \frac{m^p + M^p}{2} \langle x, y \rangle - \langle A^p x, y \rangle \right| \tag{4.111}$$

$$\leq \left\langle \left| A^p - \frac{m^p + M^p}{2} \cdot 1_H \right| x, x \right\rangle^{1/2} \left\langle \left| A^p - \frac{m^p + M^p}{2} \cdot 1_H \right| y, y \right\rangle^{1/2}$$

$$\leq \frac{1}{2} \left(M^p - m^p \right) \|x\| \, \|y\| \,,$$

and, for $m > 0$,

$$\left| \frac{m^p + M^p}{2 m^p M^p} \langle x, y \rangle - \langle A^{-p} x, y \rangle \right| \tag{4.112}$$

$$\leq \left\langle \left| A^{-p} - \frac{m^p + M^p}{2 m^p M^p} \cdot 1_H \right| x, x \right\rangle^{1/2} \left\langle \left| A^{-p} - \frac{m^p + M^p}{2 m^p M^p} \cdot 1_H \right| y, y \right\rangle^{1/2}$$

$$\leq \frac{1}{2} \left(\frac{M^p - m^p}{M^p m^p} \right) \|x\| \, \|y\| \,,$$

for any $x, y \in H$.

The proof follows from Corollary 293.

4.6.2   Logarithmic Inequalities

Consider the function $f(t) = \ln t, t > 0$. Denote by $A(a, b) := \frac{a+b}{2}$ the arithmetic mean of $a, b > 0$ and $G(a, b) := \sqrt{ab}$ the geometric mean of these numbers. We have the following result:

Proposition 296 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 < m < M$. For any $x, y \in H$ we have*

$$|\ln A(m, M) \cdot \langle x, y \rangle - \langle \ln Ax, y \rangle| \tag{4.113}$$

$$\leq \frac{1}{m} \left\langle \left| A - \frac{m + M}{2} \cdot 1_H \right| x, x \right\rangle^{1/2} \left\langle \left| A - \frac{m + M}{2} \cdot 1_H \right| y, y \right\rangle^{1/2}$$

$$\leq \frac{1}{2} \left(\frac{M}{m} - 1 \right) \|x\| \, \|y\|$$

and

$$|\ln G(m, M) \cdot \langle x, y \rangle - \langle \ln Ax, y \rangle| \tag{4.114}$$

$$\leq \langle |\ln A - \ln G(m, M) \cdot 1_H| x, x \rangle^{1/2} \langle |\ln A - \ln G(m, M) \cdot 1_H| y, y \rangle^{1/2}$$

$$\leq \ln \sqrt{\frac{M}{m}} \cdot \|x\| \, \|y\| \,.$$

The proof follows by (4.101) and (4.108).

4.7   Ostrowski's Type Vector Inequalities

4.7.1   Some Vector Inequalities

The following result holds:

Theorem 297 (Dragomir, 2010, [26]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\left| \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s)\, ds - \langle f(A) x, y \rangle \right| \tag{4.115}$$

$$\leq \frac{1}{M - m} \bigvee_m^M (f) \max_{t \in [m, M]} \left[(M - t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right.$$

$$\left. + (t - m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \right]$$

$$\leq \|x\| \, \|y\| \bigvee_m^M (f)$$

for any $x, y \in H$.

Proof. Assume that $f : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$. Then under the assumptions of the theorem for A and $\{E_\lambda\}_\lambda$, we have the following representation

$$\langle x, y \rangle \frac{1}{M - m} \int_m^M f(s)\, ds - \langle f(A) x, y \rangle \tag{4.116}$$

$$= \frac{1}{M - m} \int_{m-0}^M \langle [(M - t) E_t + (t - m)(E_t - 1_H)] x, y \rangle\, df(t)$$

for any $x, y \in H$.

Indeed, integrating by parts in the Riemann-Stieltjes integral and using the spectral representation theorem we have

$$
\frac{1}{M-m} \int_{m-0}^{M} \langle [(M-t) E_t + (t-m)(E_t - 1_H)] x, y \rangle \, df(t)
$$

$$
= \int_{m-0}^{M} \left(\langle E_t x, y \rangle - \frac{t-m}{M-m} \langle x, y \rangle \right) df(t)
$$

$$
= \left(\langle E_t x, y \rangle - \frac{t-m}{M-m} \langle x, y \rangle \right) f(t) \Big|_{m-0}^{M}
$$

$$
- \int_{m-0}^{M} f(t) \, d \left(\langle E_t x, y \rangle - \frac{t-m}{M-m} \langle x, y \rangle \right)
$$

$$
= - \int_{m-0}^{M} f(t) \, d \langle E_t x, y \rangle + \langle x, y \rangle \frac{1}{M-m} \int_{m}^{M} f(t) \, dt
$$

$$
= \langle x, y \rangle \frac{1}{M-m} \int_{m}^{M} f(t) \, dt - \langle f(A) x, y \rangle
$$

for any $x, y \in H$ and the equality (4.116) is proved.

It is well known that if $p : [a,b] \to \mathbb{C}$ is a continuous function and $v : [a,b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$
\left| \int_a^b p(t) \, dv(t) \right| \leq \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v)
$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a,b]$.

Utilising this property we have from (4.116) that

$$
\left| \langle x, y \rangle \frac{1}{M-m} \int_{m}^{M} f(s) \, ds - \langle f(A) x, y \rangle \right| \tag{4.117}
$$

$$
\leq \frac{1}{M-m} \max_{t \in [m,M]} |\langle [(M-t) E_t + (t-m)(E_t - 1_H)] x, y \rangle| \bigvee_m^M (f)
$$

for any $x, y \in H$.

Now observe that

$$
\begin{aligned}
&|\langle [(M-t) E_t + (t-m)(E_t - 1_H)] x, y \rangle| \tag{4.118}\\
&= |(M-t) \langle E_t x, y \rangle + (t-m) \langle (E_t - 1_H) x, y \rangle|\\
&\leq (M-t) |\langle E_t x, y \rangle| + (t-m) |\langle (E_t - 1_H) x, y \rangle|
\end{aligned}
$$

for any $x, y \in H$ and $t \in [m, M]$.

If P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle \tag{4.119}$$

for any $x, y \in H$.

On applying the inequality (4.119) we have

$$
(M - t) |\langle E_t x, y\rangle| + (t - m) |\langle (E_t - 1_H) x, y\rangle| \tag{4.120}
$$
$$
\leq (M - t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2}
$$
$$
+ (t - m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2}
$$
$$
\leq \max\{M - t, t - m\}
$$
$$
\times \left[\langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} + \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \right]
$$
$$
\leq \max\{M - t, t - m\}
$$
$$
\times [\langle E_s x, x\rangle + \langle (1_H - E_s) x, x\rangle]^{1/2} [\langle E_s y, y\rangle + \langle (1_H - E_s) y, y\rangle]^{1/2}
$$
$$
= \max\{M - t, t - m\} \|x\| \|y\|,
$$

where for the last inequality we used the elementary fact

$$a_1 b_1 + a_2 b_2 \leq \left(a_1^2 + a_2^2\right)^{1/2} \left(b_1^2 + b_2^2\right)^{1/2} \tag{4.121}$$

that holds for a_1, b_1, a_2, b_2 positive real numbers.

Utilising the inequalities (4.117), (4.118) and (4.120) we deduce the desired result (4.115). ∎

The case of Lipschitzian functions is embodied in the following result:

Theorem 298 (Dragomir, 2010, [26]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$
\left| \langle x, y\rangle \frac{1}{M - m} \int_m^M f(s)\, ds - \langle f(A) x, y\rangle \right| \tag{4.122}
$$
$$
\leq \frac{L}{M - m} \int_m^M \left[(M - t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \right.
$$
$$
\left. + (t - m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \right] dt
$$
$$
\leq \frac{3}{4} L (M - m) \|x\| \|y\|
$$

for any $x, y \in H$.

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\, dv(t) \right| \leq L \int_a^b |p(t)|\, dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (4.116) that

$$\left| \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s)\, ds - \langle f(A)x, y \rangle \right| \tag{4.123}$$

$$\leq \frac{L}{M - m} \int_{m-0}^M |\langle [(M - t)E_t + (t - m)(E_t - 1_H)]x, y \rangle|\, dt.$$

Since, from the proof of Theorem 297, we have

$$|\langle [(M - t)E_t + (t - m)(E_t - 1_H)]x, y \rangle| \tag{4.124}$$

$$\leq (M - t)\langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2}$$

$$+ (t - m)\langle (1_H - E_t)x, x \rangle^{1/2} \langle (1_H - E_t)y, y \rangle^{1/2}$$

$$\leq \max\{M - t, t - m\} \|x\| \|y\|$$

$$= \left[\frac{1}{2}(M - m) + \left| t - \frac{m + M}{2} \right| \right] \|x\| \|y\|$$

for any $x, y \in H$ and $t \in [m, M]$, then integrating (4.124) and taking into account that

$$\int_m^M \left| t - \frac{m + M}{2} \right|\, dt = \frac{1}{4}(M - m)^2$$

we deduce the desired result (4.122). ■

Finally for the section, we provide here the case of monotonic nondecreasing functions as well:

Theorem 299 (Dragomir, 2010, [26]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on $[m, M]$, then we have*

the inequality

$$\left| \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s) \, ds - \langle f(A) x, y \rangle \right| \tag{4.125}$$

$$\leq \frac{1}{M - m} \int_m^M \left[(M - t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right.$$

$$+ (t - m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \Big] \, df(t)$$

$$\leq \left[f(M) - f(m) - \frac{1}{M - m} \int_m^M sgn \left(t - \frac{m + M}{2} \right) f(t) \, dt \right] \|x\| \, \|y\|$$

$$\leq [f(M) - f(m)] \|x\| \, \|y\|$$

for any $x, y \in H$.

Proof. From the theory of Riemann-Stieltjes integral is well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t) \, dv(t)$ and $\int_a^b |p(t)| \, dv(t)$ exist and

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \int_a^b |p(t)| \, dv(t).$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (4.116) that

$$\left| \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s) \, ds - \langle f(A) x, y \rangle \right| \tag{4.126}$$

$$\leq \frac{1}{M - m} \int_{m-0}^M |\langle [(M - t) E_t + (t - m)(E_t - 1_H)] x, y \rangle| \, df(t).$$

Further on, by utilizing the inequality (4.124) we also have that

$$\int_{m-0}^M |\langle [(M - t) E_t + (t - m)(E_t - 1_H)] x, y \rangle| \, df(t) \tag{4.127}$$

$$\leq \int_m^M \left[(M - t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right.$$

$$+ (t - m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \Big] \, df(t)$$

$$\leq \left[\frac{1}{2} (M - m) [f(M) - f(m)] + \int_m^M \left| t - \frac{m + M}{2} \right| df(t) \right] \|x\| \, \|y\|.$$

Now, integrating by parts in the Riemann-Stieltjes integral we have

$$
\int_m^M \left| t - \frac{m+M}{2} \right| df(t)
$$
$$
= \int_m^{\frac{M+m}{2}} \left(\frac{m+M}{2} - t \right) df(t) + \int_{\frac{m+M}{2}}^M \left(t - \frac{m+M}{2} \right) df(t)
$$
$$
= \left(\frac{m+M}{2} - t \right) f(t) \Big|_m^{\frac{M+m}{2}} + \int_m^{\frac{M+m}{2}} f(t) \, dt
$$
$$
+ \left(t - \frac{m+M}{2} \right) f(t) \Big|_{\frac{m+M}{2}}^M - \int_{\frac{m+M}{2}}^M f(t) \, dt
$$
$$
= \frac{1}{2}(M-m)[f(M) - f(m)] - \int_m^M sgn\left(t - \frac{m+M}{2} \right) f(t) \, dt,
$$

which together with (4.127) produces the second inequality in (4.125).

Since the functions $sgn\left(\cdot - \frac{m+M}{2} \right)$ and $f(\cdot)$ have the same monotonicity, then by the Čebyšev inequality we have

$$
\int_m^M sgn\left(t - \frac{m+M}{2} \right) f(t) \, dt
$$
$$
\geq \frac{1}{M-m} \int_m^M sgn\left(t - \frac{m+M}{2} \right) dt \int_m^M f(t) \, dt = 0
$$

and the last part of (4.125) is proved. ∎

4.7.2 Applications for Particular Functions

It is obvious that the above results can be applied for various particular functions. However, we will restrict here only to the power and logarithmic functions.

1. Consider now the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^p$ with $p > 0$. This function is monotonic increasing on $(0, \infty)$ and applying Theorem 299 we can state the following proposition:

Proposition 300 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the*

inequalities

$$\left| \langle A^p x, y \rangle - \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)} \langle x, y \rangle \right| \tag{4.128}$$

$$\leq \frac{p}{M-m} \int_m^M \left[(M-t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right.$$

$$\left. + (t-m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \right] t^{p-1} dt$$

$$\leq \left[M^p - m^p - \frac{M^{p+1} + m^{p+1} - 2^p (M+m)^{p+1}}{(p+1)(M-m)} \right] \|x\| \|y\|.$$

On applying now Theorem 298 to the same power function, then we can state the following result as well:

Proposition 301 *With the same assumptions from Proposition 300 we have*

$$\left| \langle A^p x, y \rangle - \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)} \langle x, y \rangle \right| \tag{4.129}$$

$$\leq \frac{B_p}{M-m} \int_m^M \left[(M-t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right.$$

$$\left. + (t-m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \right] dt$$

$$\leq \frac{3}{4} B_p (M-m) \|x\| \|y\|$$

for any $x, y \in H$, where

$$B_p = p \times \begin{cases} M^{p-1} & \text{if } p \geq 1 \\ \\ m^{p-1} & \text{if } 0 < p < 1, m > 0. \end{cases}$$

The case of negative powers except $p = -1$ goes likewise and we omit the details.

Now, if we apply Theorem 299 and 298 for the increasing function $f(t) = -\frac{1}{t}$ with $t > 0$, then we can state the following proposition:

Proposition 302 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the*

inequalities

$$\left| \langle A^{-1}x, y \rangle - \frac{\ln M - \ln m}{M - m} \langle x, y \rangle \right| \tag{4.130}$$

$$\leq \frac{1}{M - m} \int_m^M \left[(M - t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right.$$

$$\left. + (t - m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \right] t^2 dt$$

$$\leq \left[\frac{M - m}{mM} - \frac{\ln \left[\left(\frac{m+M}{2} \right)^2 \right] - \ln (mM)}{M - m} \right] \|x\| \|y\|$$

and

$$\left| \langle A^{-1}x, y \rangle - \frac{\ln M - \ln m}{M - m} \langle x, y \rangle \right| \tag{4.131}$$

$$\leq \frac{1}{m^2 (M - m)} \int_m^M \left[(M - t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right.$$

$$\left. + (t - m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \right] dt$$

$$\leq \frac{3}{4} \frac{M - m}{m^2} \|x\| \|y\|.$$

2. Now, if we apply Theorems 299 and 298 to the function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then we can state

Proposition 303 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$|\langle \ln Ax, y \rangle - \langle x, y \rangle \ln I(m, M)| \tag{4.132}$$

$$\leq \frac{1}{M - m} \int_m^M \left[(M - t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right.$$

$$\left. + (t - m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \right] t dt$$

$$\leq \left[\ln \left(\frac{M}{m} \right) - \ln \left(\sqrt{\frac{I \left(\frac{m+M}{2}, M \right)}{I \left(m, \frac{m+M}{2} \right)}} \right) \right] \|x\| \|y\|$$

and

$$|\langle \ln Ax, y \rangle - \langle x, y \rangle \ln I(m, M)| \tag{4.133}$$

$$\leq \frac{1}{m(M-m)} \int_m^M \left[(M-t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right. $$

$$\left. + (t-m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \right] dt$$

$$\leq \frac{3}{4} \left(\frac{M}{m} - 1 \right) \|x\| \|y\|,$$

where $I(m, M)$ is the identric mean of m and M and is defined by

$$I(m, M) = \frac{1}{e} \left(\frac{M^M}{m^m} \right)^{1/(M-m)}.$$

4.8   Bounds for the Difference Between Functions and Integral Means

4.8.1   Vector Inequalities Via Ostrowski's Type Bounds

The following result holds:

Theorem 304 (Dragomir, 2010, [22]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous function on $[m, M]$, then we have the inequality*

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M-m} \int_m^M f(s) \, ds \right| \tag{4.134}$$

$$\leq \max_{t \in [m,M]} \left| f(t) - \frac{1}{M-m} \int_m^M f(s) \, ds \right| \bigvee_m^M (\langle E_{(.)} x, y \rangle)$$

$$\leq \max_{t \in [m,M]} \left| f(t) - \frac{1}{M-m} \int_m^M f(s) \, ds \right| \|x\| \|y\|$$

for any $x, y \in H$.

Proof. Utilising the spectral representation theorem we have the following equality of interest

$$\langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M-m} \int_m^M f(s) \, ds \tag{4.135}$$

$$= \int_{m-0}^M \left[f(t) - \frac{1}{M-m} \int_m^M f(s) \, ds \right] d(\langle E_t x, y \rangle)$$

for any $x, y \in H$.

It is well known that if $p : [a, b] \rightarrow \mathbb{C}$ is a continuous function and $v : [a, b] \rightarrow \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v), \qquad (4.136)$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising these two facts we get the first part of (4.134).

The last part follows by the Total Variation Schwarz's inequality and we omit the details. ∎

For particular classes of continuous functions $f : [m, M] \rightarrow \mathbb{C}$ we are able to provide simpler bounds as incorporated in the following corollary:

Corollary 305 (Dragomir, 2010, [22]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family and $f : [m, M] \rightarrow \mathbb{C}$ a continuous function on $[m, M]$.*

1. If f is of bounded variation on $[m, M]$, then

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s) \, ds \right| \qquad (4.137)$$

$$\leq \bigvee_m^M (f) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \|x\| \, \|y\| \bigvee_m^M (f)$$

for any $x, y \in H$.

2. If $f : [m, M] \longrightarrow \mathbb{C}$ is of $r - H - Hölder$ type, i.e., for a given $r \in (0, 1]$ and $H > 0$ we have

$$|f(s) - f(t)| \leq H \, |s - t|^r \quad \text{for any } s, t \in [m, M], \qquad (4.138)$$

then we have the inequality:

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s) \, ds \right| \qquad (4.139)$$

$$\leq \frac{1}{r + 1} H (M - m)^r \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \frac{1}{r + 1} H (M - m)^r \|x\| \, \|y\|$$

for any $x, y \in H$.

In particular, if $f : [m, M] \longrightarrow \mathbb{C}$ is Lipschitzian with the constant $L > 0$, then

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s) \, ds \right| \tag{4.140}$$

$$\leq \frac{1}{2} L (M - m) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \frac{1}{2} L (M - m) \| x \| \| y \|$$

for any $x, y \in H$.

3. *If $f : [m, M] \longrightarrow \mathbb{C}$ is absolutely continuous, then*

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s) \, ds \right| \tag{4.141}$$

$$\leq \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\times \begin{cases} \frac{1}{2} (M - m) \| f' \|_\infty & \text{if } f' \in L_\infty [m, M] \\[2mm] \frac{1}{(q+1)^{1/q}} (M - m)^{1/q} \| f' \|_p & \begin{array}{l} \text{if } f' \in L_p [m, M] \\ p > 1, 1/p + 1/q = 1; \end{array} \\[2mm] \| f' \|_1 \end{cases}$$

$$\leq \| x \| \| y \|$$

$$\times \begin{cases} \frac{1}{2} (M - m) \| f' \|_\infty & \text{if } f' \in L_\infty [m, M] \\[2mm] \frac{1}{(q+1)^{1/q}} (M - m)^{1/q} \| f' \|_p & \begin{array}{l} \text{if } f' \in L_p [m, M] \\ p > 1, 1/p + 1/q = 1; \end{array} \\[2mm] \| f' \|_1 \end{cases}$$

for any $x, y \in H$, where $\| f' \|_p$ are the Lebesgue norms, i.e., we recall that

$$\| f' \|_p := \begin{cases} \operatorname{ess\,sup}_{s \in [m, M]} |f'(s)| & \text{if } p = \infty; \\[2mm] \left(\int_m^M |f(s)|^p \, ds \right)^{1/p} & \text{if } p \geq 1. \end{cases}$$

Proof. We use the Ostrowski type inequalities in order to provide upper bounds for the quantity

$$\max_{t \in [m, M]} \left| f(t) - \frac{1}{M - m} \int_m^M f(s) \, ds \right|$$

where $f : [m, M] \longrightarrow \mathbb{C}$ is a continuous function.

The following result may be stated (see [23]) for functions of bounded variation:

Lemma 306 *Assume that* $f : [m, M] \to \mathbb{C}$ *is of bounded variation and denote by* $\bigvee_{m}^{M} (f)$ *its total variation. Then*

$$\left| f(t) - \frac{1}{M - m} \int_{m}^{M} f(s)\, ds \right| \tag{4.142}$$

$$\leq \left[\frac{1}{2} + \left| \frac{t - \frac{m+M}{2}}{M - m} \right| \right] \bigvee_{m}^{M} (f)$$

for all $t \in [m, M]$. *The constant* $\frac{1}{2}$ *is the best possible.*

Now, taking the maximum over $x \in [m, M]$ in (4.142) we deduce (4.137). If f is Hölder continuous, then one may state the result:

Lemma 307 *Let* $f : [m, M] \to \mathbb{C}$ *be of* $r - H - H\ddot{o}lder$ *type, where* $r \in (0, 1]$ *and* $H > 0$ *are fixed, then, for all* $x \in [m, M]$, *we have the inequality:*

$$\left| f(t) - \frac{1}{M - m} \int_{m}^{M} f(s)\, ds \right| \tag{4.143}$$

$$\leq \frac{H}{r + 1} \left[\left(\frac{M - t}{M - m} \right)^{r+1} + \left(\frac{t - m}{M - m} \right)^{r+1} \right] (M - m)^{r}.$$

The constant $\frac{1}{r+1}$ *is also sharp in the above sense.*

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [17])

$$\left| f(t) - \frac{1}{M - m} \int_{m}^{M} f(s)\, ds \right| \tag{4.144}$$

$$\leq \left[\frac{1}{4} + \left(\frac{t - \frac{m+M}{2}}{M - m} \right)^{2} \right] (M - m) L,$$

for any $x \in [m, M]$. Here the constant $\frac{1}{4}$ is also best.

Taking the maximum over $x \in [m, M]$ in (4.143) we deduce (4.139) and the second part of the corollary is proved.

The following Ostrowski type result for absolutely continuous functions holds.

Lemma 308 *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. Then, for all $t \in [a, b]$, we have:*

$$\left| f\left(t\right) - \frac{1}{M - m} \int_m^M f\left(s\right) ds \right| \tag{4.145}$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{t - \frac{m+M}{2}}{M-m} \right)^2 \right] \left(M - m\right) \|f'\|_\infty & \text{if} \quad f' \in L_\infty\left[m, M\right]; \\[4mm] \frac{1}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{t-m}{M-m} \right)^{q+1} + \left(\frac{M-t}{M-m} \right)^{q+1} \right]^{\frac{1}{q}} \left(M - m\right)^{\frac{1}{q}} \|f'\|_p & \text{if} \quad f' \in L_p\left[m, M\right], \\[1mm] & \frac{1}{p} + \frac{1}{q} = 1, \ p > 1; \\[4mm] \left[\frac{1}{2} + \left| \frac{t - \frac{m+M}{2}}{M-m} \right| \right] \|f'\|_1 . \end{cases}$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented above.

The above inequalities can also be obtained from the Fink result in [39] on choosing $n = 1$ and performing some appropriate computations.

Taking the maximum in these inequalities we deduce (4.141). ∎

For other scalar Ostrowski's type inequalities, see [1] and [18].

4.8.2   *Other Vector Inequalities*

In [37], the authors have considered the following functional

$$D\left(f; u\right) := \int_a^b f\left(s\right) du\left(s\right) - \left[u\left(b\right) - u\left(a\right)\right] \cdot \frac{1}{b - a} \int_a^b f\left(t\right) dt, \tag{4.146}$$

provided that the Stieltjes integral $\int_a^b f\left(s\right) du\left(s\right)$ exists.

This functional plays an important role in approximating the Stieltjes integral $\int_a^b f\left(s\right) du\left(s\right)$ in terms of the Riemann integral $\int_a^b f\left(t\right) dt$ and the divided difference of the integrator u.

In [37], the following result in estimating the above functional $D\left(f; u\right)$ has been obtained:

$$\left|D\left(f; u\right)\right| \leq \frac{1}{2} L\left(M - m\right)\left(b - a\right), \tag{4.147}$$

provided u is $L-Lipschitzian$ and f is *Riemann integrable* and with the property that there exists the constants $m, M \in \mathbb{R}$ such that

$$m \leq f\left(t\right) \leq M \quad \text{for any} \quad t \in [a, b]. \tag{4.148}$$

The constant $\frac{1}{2}$ is best possible in (4.147) in the sense that it cannot be replaced by a smaller quantity.

If one assumes that u is of *bounded variation* and f is $K-Lipschitzian$, then $D(f, u)$ satisfies the inequality [38]

$$|D(f; u)| \leq \frac{1}{2} K (b - a) \bigvee_{a}^{b} (u).$$ (4.149)

Here the constant $\frac{1}{2}$ is also best possible.

Now, for the function $u : [a, b] \rightarrow \mathbb{C}$, consider the following auxiliary mappings Φ, Γ and Δ [19]:

$$\Phi(t) := \frac{(t - a) u(b) + (b - t) u(a)}{b - a} - u(t), \qquad t \in [a, b],$$

$$\Gamma(t) := (t - a) [u(b) - u(t)] - (b - t) [u(t) - u(a)], \qquad t \in [a, b],$$

$$\Delta(t) := [u; b, t] - [u; t, a], \qquad t \in (a, b),$$

where $[u; \alpha, \beta]$ is the *divided difference* of u in α, β, i.e.,

$$[u; \alpha, \beta] := \frac{u(\alpha) - u(\beta)}{\alpha - \beta}.$$

The following representation of $D(f, u)$ may be stated, see [19] and [20]. Due to its importance in proving our new results we present here a short proof as well.

Lemma 309 *Let $f, u : [a, b] \rightarrow \mathbb{C}$ be such that the Stieltjes integral $\int_a^b f(t) \, du(t)$ and the Riemann integral $\int_a^b f(t) \, dt$ exist. Then*

$$D(f, u) = \int_a^b \Phi(t) \, df(t) = \frac{1}{b - a} \int_a^b \Gamma(t) \, df(t)$$ (4.150)

$$= \frac{1}{b - a} \int_a^b (t - a)(b - t) \Delta(t) \, df(t).$$

Proof. Since $\int_a^b f(t) \, du(t)$ exists, hence $\int_a^b \Phi(t) \, df(t)$ also exists, and the integration by parts formula for Riemann-Stieltjes integrals gives that

$$\int_a^b \Phi(t) \, df(t) = \int_a^b \left[\frac{(t - a) u(b) + (b - t) u(a)}{b - a} - u(t) \right] df(t)$$

$$= \left[\frac{(t - a) u(b) + (b - t) u(a)}{b - a} - u(t) \right] f(t) \Big|_a^b$$

$$- \int_a^b f(t) \, d \left[\frac{(t - a) u(b) + (b - t) u(a)}{b - a} - u(t) \right]$$

$$= - \int_a^b f(t) \left[\frac{u(b) - u(a)}{b - a} dt - du(t) \right] = D(f, u),$$

proving the required identity. ∎

For recent inequalities related to $D(f; u)$ for various pairs of functions (f, u), see [21].

The following representation for a continuous function of selfadjoint operator may be stated:

Lemma 310 (Dragomir, 2010, [22]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family and $f : [m, M] \to \mathbb{C}$ a continuous function on $[m, M]$. If $x, y \in H$, then we have the representation*

$$\langle f(A) x, y \rangle = \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s) \, ds \tag{4.151}$$

$$+ \frac{1}{M - m} \int_{m-0}^M \langle [(t - m)(1_H - E_t) - (M - t) E_t] x, y \rangle \, df(t).$$

Proof. Utilising Lemma 309 we have

$$\int_m^M f(t) \, du(t) = [u(M) - u(m)] \cdot \frac{1}{M - m} \int_m^M f(s) \, ds \tag{4.152}$$

$$+ \int_m^M \left[\frac{(t - m) u(M) + (M - t) u(m)}{M - m} - u(t) \right] df(t),$$

for any continuous function $f : [m, M] \to \mathbb{C}$ and any function of bounded variation $u : [m, M] \to \mathbb{C}$.

Now, if we write the equality (4.152) for $u(t) = \langle E_t x, y \rangle$ with $x, y \in H$, then we get

$$\int_{m-0}^M f(t) \, d\langle E_t x, y \rangle = \langle x, y \rangle \cdot \frac{1}{M - m} \int_m^M f(s) \, ds \tag{4.153}$$

$$+ \int_{m-0}^M \left[\frac{(t - m) \langle x, y \rangle}{M - m} - \langle E_t x, y \rangle \right] df(t),$$

which, by the spectral representation theorem, produces the desired result (4.151). ∎

The following result may be stated:

Theorem 311 (Dragomir, 2010, [22]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ $\{E_\lambda\}_\lambda$ be its spectral family and $f : [m, M] \to \mathbb{C}$ a continuous function on $[m, M]$.*

1. *If f is of bounded variation, then*

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M-m} \int_m^M f(s) \, ds \right| \tag{4.154}$$

$$\leq \|y\| \bigvee_m^M (f)$$

$$\times \max_{t \in [m,M]} \left[\left(\frac{t-m}{M-m} \right)^2 \|(1_H - E_t) x\|^2 + \left(\frac{M-t}{M-m} \right)^2 \|E_t x\|^2 \right]^{1/2}$$

$$\leq \|x\| \, \|y\| \bigvee_m^M (f)$$

for any $x, y \in H$.

2. *If f is Lipschitzian with the constant $L > 0$, then*

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M-m} \int_m^M f(s) \, ds \right| \tag{4.155}$$

$$\leq \frac{L \|y\|}{M-m} \int_{m-0}^M \left[(t-m)^2 \|(1_H - E_t) x\|^2 + (M-t)^2 \|E_t x\|^2 \right]^{1/2} dt$$

$$\leq \frac{1}{2} \left[1 + \frac{\sqrt{2}}{2} \ln \left(\sqrt{2} + 1 \right) \right] (M-m) L \|y\| \, \|x\|$$

for any $x, y \in H$.

3. *If $f : [m, M] \to \mathbb{R}$ is monotonic nondecreasing, then*

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M-m} \int_m^M f(s) \, ds \right| \tag{4.156}$$

$$\leq \frac{\|y\|}{M-m} \int_{m-0}^M \left[(t-m)^2 \|(1_H - E_t) x\|^2 + (M-t)^2 \|E_t x\|^2 \right]^{1/2} df(t)$$

$$\leq \|y\| \, \|x\| \int_m^M \left[\left(\frac{t-m}{M-m} \right)^2 + \left(\frac{M-t}{M-m} \right)^2 \right]^{1/2} df(t)$$

$$\leq \|y\| \, \|x\| \, [f(M) - f(m)]^{1/2}$$

$$\times \left[f(M) - f(m) - \frac{4}{M-m} \int_m^M \left(t - \frac{m+M}{2} \right) f(t) \, dt \right]^{1/2}$$

$$\leq \|y\| \, \|x\| \, [f(M) - f(m)]$$

for any $x, y \in H$.

Proof. If we assume that f is of bounded variation, then on applying the property (4.136) to the representation (4.151) we get

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s)\, ds \right| \tag{4.157}$$

$$\leq \frac{1}{M - m} \max_{t \in [m, M]} |\langle [(t - m)(1_H - E_t) - (M - t) E_t] x, y \rangle| \bigvee_m^M (f).$$

Now, on utilizing the Schwarz inequality and the fact that E_t is a projector for any $t \in [m, M]$, then we have

$$|\langle [(t - m)(1_H - E_t) - (M - t) E_t] x, y \rangle| \tag{4.158}$$
$$\leq \| [(t - m)(1_H - E_t) - (M - t) E_t] x \| \, \|y\|$$
$$= \left[(t - m)^2 \|(1_H - E_t) x\|^2 + (M - t)^2 \|E_t x\|^2 \right]^{1/2} \|y\|$$
$$\leq \left[(t - m)^2 + (M - t)^2 \right]^{1/2} \|x\| \, \|y\|$$

for any $x, y \in H$ and for any $t \in [m, M]$.

Taking the maximum in (4.158) we deduce the desired inequality (4.154).

It is well known that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\, dv(t) \right| \leq L \int_a^b |p(t)|\, dt.$$

Now, on applying this property of the Riemann-Stieltjes integral to the representation (4.151), we get

$$\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s)\, ds \right| \tag{4.159}$$

$$\leq \frac{L}{M - m} \int_{m-0}^M |\langle [(t - m)(1_H - E_t) - (M - t) E_t] x, y \rangle|\, dt$$

$$\leq \frac{L \|y\|}{M - m} \int_{m-0}^M \left[(t - m)^2 \|(1_H - E_t) x\|^2 + (M - t)^2 \|E_t x\|^2 \right]^{1/2} dt$$

$$\leq L \|y\| \, \|x\| \int_m^M \left[\left(\frac{t - m}{M - m} \right)^2 + \left(\frac{M - t}{M - m} \right)^2 \right]^{1/2} dt,$$

for any $x, y \in H$.

Now, if we change the variable in the integral by choosing $u = \frac{t-m}{M-m}$ then we get

$$
\int_m^M \left[\left(\frac{t-m}{M-m} \right)^2 + \left(\frac{M-t}{M-m} \right)^2 \right]^{1/2} dt
$$

$$
= (M-m) \int_0^1 \left[u^2 + (1-u)^2 \right]^{1/2} du
$$

$$
= \frac{1}{2} (M-m) \left[1 + \frac{\sqrt{2}}{2} \ln \left(\sqrt{2} + 1 \right) \right],
$$

which together with (4.159) produces the desired result (4.155).

From the theory of Riemann-Stieltjes integral is well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t) \, dv(t)$ and $\int_a^b |p(t)| \, dv(t)$ exist and

$$
\left| \int_a^b p(t) \, dv(t) \right| \leq \int_a^b |p(t)| \, dv(t).
$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (4.151)

$$
\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M-m} \int_m^M f(s) \, ds \right| \tag{4.160}
$$

$$
\leq \frac{1}{M-m} \int_{m-0}^M \left| \langle [(t-m)(1_H - E_t) - (M-t) E_t] x, y \rangle \right| df(t)
$$

$$
\leq \frac{\|y\|}{M-m} \int_{m-0}^M \left[(t-m)^2 \|(1_H - E_t) x\|^2 + (M-t)^2 \|E_t x\|^2 \right]^{1/2} df(t)
$$

$$
\leq \|y\| \, \|x\| \int_m^M \left[\left(\frac{t-m}{M-m} \right)^2 + \left(\frac{M-t}{M-m} \right)^2 \right]^{1/2} df(t),
$$

for any $x, y \in H$ and the proof of the first and second inequality in (4.156) is completed.

For the last part we use the following Cauchy-Buniakowski-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrator v

$$
\left| \int_a^b p(t) q(t) \, dv(t) \right| \leq \left[\int_a^b |p(t)|^2 \, dv(t) \right]^{1/2} \left[\int_a^b |q(t)|^2 \, dv(t) \right]^{1/2}
$$

where $p, q : [a, b] \to \mathbb{C}$ are continuous on $[a, b]$.

By applying this inequality we conclude that

$$\int_m^M \left[\left(\frac{t-m}{M-m} \right)^2 + \left(\frac{M-t}{M-m} \right)^2 \right]^{1/2} df(t) \qquad (4.161)$$

$$\leq \left[\int_m^M df(t) \right]^{1/2} \left[\int_m^M \left[\left(\frac{t-m}{M-m} \right)^2 + \left(\frac{M-t}{M-m} \right)^2 \right] df(t) \right]^{1/2}.$$

Further, integrating by parts in the Riemann-Stieltjes integral we also have that

$$\int_m^M \left[\left(\frac{t-m}{M-m} \right)^2 + \left(\frac{M-t}{M-m} \right)^2 \right] df(t) \qquad (4.162)$$

$$= f(M) - f(m) - \frac{4}{M-m} \int_m^M \left(t - \frac{m+M}{2} \right) f(t)\, dt$$

$$\leq f(M) - f(m)$$

where for the last part we used the fact that by the Čebyšev integral inequality for monotonic functions with the same monotonicity we have that

$$\int_m^M \left(t - \frac{m+M}{2} \right) f(t)\, dt$$

$$\geq \frac{1}{M-m} \int_m^M \left(t - \frac{m+M}{2} \right) dt \int_m^M f(t)\, dt = 0.$$

∎

4.8.3 Some Applications for Particular Functions

1. Consider the function $f : (0, \infty) \to \mathbb{R}$ given by $f(t) = t^r$ with $r \in (0, 1]$. This function is r-Hölder continuous with the constant $H > 0$. Then, by applying Corollary 305 we can state the following result

Proposition 312 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family. Then for all r with $r \in (0, 1]$ we have the inequality*

$$\left| \langle A^r x, y \rangle - \langle x, y \rangle \frac{M^{r+1} - m^{r+1}}{(r+1)(M-m)} \right| \qquad (4.163)$$

$$\leq \frac{1}{r+1} (M-m)^r \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \frac{1}{r+1} (M-m)^r \|x\| \|y\|$$

for any $x, y \in H$.

The case of $p > 1$ is incorporated in the following proposition:

Proposition 313 *With the same assumptions from Proposition 312 and if $p > 1$, then we have*

$$\left| \langle A^p x, y \rangle - \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)} \langle x, y \rangle \right| \tag{4.164}$$

$$\leq \frac{1}{2} p M^{p-1} (M-m) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \frac{1}{2} p M^{p-1} (M-m) \|x\| \|y\|$$

for any $x, y \in H$.

The case of negative powers except $p = -1$ goes likewise and we omit the details.

Now, if we apply Corollary 305 for the function $f(t) = -\frac{1}{t}$ with $t > 0$, then we can state the following proposition:

Proposition 314 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\left| \langle A^{-1} x, y \rangle - \frac{\ln M - \ln m}{M - m} \langle x, y \rangle \right| \tag{4.165}$$

$$\leq \frac{1}{2} \frac{M - m}{m^2} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \frac{1}{2} \frac{M - m}{m^2} \|x\| \|y\|.$$

2. Now, if we apply Corollary 305 to the function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then we can state

Proposition 315 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\left| \langle \ln A x, y \rangle - \langle x, y \rangle \ln I(m, M) \right| \tag{4.166}$$

$$\leq \frac{1}{2} \left(\frac{M}{m} - 1 \right) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \frac{1}{2} \left(\frac{M}{m} - 1 \right) \|x\| \|y\|,$$

where $I(m, M)$ is the identric mean of m and M and is defined by

$$I(m, M) = \frac{1}{e} \left(\frac{M^M}{m^m} \right)^{1/(M-m)}.$$

4.9   Ostrowski's Type Inequalities for n-Time Differentiable Functions

4.9.1   Some Identities

In [6], the authors have pointed out the following integral identity:

Lemma 316 (Cerone-Dragomir-Roumeliotis, 1999, [6]) *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the $(n-1)$-derivative $f^{(n-1)}$ (where $n \geq 1$) is absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$, we have the identity:*

$$\int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \qquad (4.167)$$

$$+ (-1)^n \int_a^b K_n(x, t) f^{(n)}(t)\, dt$$

where the kernel $K_n : [a, b]^2 \to \mathbb{R}$ is given by

$$K_n(x, t) := \begin{cases} \frac{(t-a)^n}{n!}, & a \leq t \leq x \leq b \\[2mm] \frac{(t-b)^n}{n!}, & a \leq x < t \leq b. \end{cases} \qquad (4.168)$$

The identity (4.168) can be written in the following equivalent form as:

$$f(z) = \frac{1}{b-a} \int_a^b f(t)\, dt \qquad (4.169)$$

$$- \frac{1}{b-a} \sum_{k=1}^{n-1} \frac{1}{(k+1)!} \left[(b-z)^{k+1} + (-1)^k (z-a)^{k+1} \right] f^{(k)}(z)$$

$$+ \frac{(-1)^{n-1}}{(b-a)\, n!} \left[\int_a^z (t-a)^n f^{(n)}(t)\, dt + \int_z^b (t-b)^n f^{(n)}(t)\, dt \right]$$

for all $z \in [a, b]$.

Note that for $n = 1$, the sum $\sum_{k=1}^{n-1}$ is empty and we obtain the well known *Montgomery's identity* (see for example [3])

$$f(z) = \frac{1}{b-a} \int_a^b f(t)\, dt \qquad (4.170)$$

$$+ \frac{1}{b-a} \left[\int_a^z (t-a) f^{(1)}(t)\, dt + \int_z^b (t-b) f^{(1)}(t)\, dt \right],$$

for any $z \in [a, b]$.

In a slightly more general setting, by the use of the identity (4.169), we can state the following result as well:

Lemma 317 (Dragomir, 2010, [8]) *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the n-derivative $f^{(n)}$ (where $n \geq 1$) is of bounded variation on $[a, b]$. Then for all $\lambda \in [a, b]$, we have the identity:*

$$f(\lambda) = \frac{1}{b-a} \int_a^b f(t)\, dt \tag{4.171}$$

$$- \frac{1}{b-a} \sum_{k=1}^n \frac{1}{(k+1)!} \left[(b-\lambda)^{k+1} + (-1)^k (\lambda - a)^{k+1} \right] f^{(k)}(\lambda)$$

$$+ \frac{(-1)^n}{(b-a)(n+1)!}$$

$$\times \left[\int_a^\lambda (t-a)^{n+1}\, d\left(f^{(n)}(t) \right) + \int_\lambda^b (t-b)^{n+1}\, d\left(f^{(n)}(t) \right) \right].$$

Now we can state the following representation result for functions of selfadjoint operators:

Theorem 318 (Dragomir, 2010, [8]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then we have the representation*

$$f(A) = \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) 1_H - \frac{1}{M-m} \tag{4.172}$$

$$\times \sum_{k=1}^n \frac{1}{(k+1)!} \left[(M 1_H - A)^{k+1} + (-1)^k (A - m 1_H)^{k+1} \right] f^{(k)}(A)$$

$$+ T_n(A, m, M)$$

where the remainder is given by

$$T_n(A, m, M) := \frac{(-1)^n}{(M-m)(n+1)!} \tag{4.173}$$

$$\times \left[\int_{m-0}^M \left(\int_m^\lambda (t-m)^{n+1}\, d\left(f^{(n)}(t) \right) \right) dE_\lambda \right.$$

$$\left. + \int_{m-0}^M \left(\int_\lambda^M (t-M)^{n+1}\, d\left(f^{(n)}(t) \right) \right) dE_\lambda \right].$$

In particular, if the n-th derivative $f^{(n)}$ is absolutely continuous on $[m, M]$, then the remainder can be represented as

$$
T_n\left(A, m, M\right) \tag{4.174}
$$
$$
= \frac{(-1)^n}{(M-m)(n+1)!}
$$
$$
\times \int_{m-0}^{M} \left[(\lambda - m)^{n+1} \left(1_H - E_\lambda\right) + (\lambda - M)^{n+1} E_\lambda\right] f^{(n+1)}\left(\lambda\right) d\lambda.
$$

Proof. By Lemma 317 we have

$$
f\left(\lambda\right) = \frac{1}{M-m} \int_m^M f\left(t\right) dt - \frac{1}{M-m} \tag{4.175}
$$
$$
\times \sum_{k=1}^{n} \frac{1}{(k+1)!} \left[(M-\lambda)^{k+1} + (-1)^k (\lambda - m)^{k+1}\right] f^{(k)}\left(\lambda\right)
$$
$$
+ \frac{(-1)^n}{(M-m)(n+1)!}
$$
$$
\times \left[\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}\left(t\right)\right) + \int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}\left(t\right)\right)\right]
$$

for any $\lambda \in [m, M]$.

Integrating the identity (4.175) in the Riemann-Stieltjes sense with the integrator E_λ we get

$$
\int_m^M f\left(\lambda\right) dE_\lambda \tag{4.176}
$$
$$
= \frac{1}{M-m} \int_m^M f\left(t\right) dt \int_m^M dE_\lambda - \frac{1}{M-m}
$$
$$
\times \sum_{k=1}^{n} \frac{1}{(k+1)!} \int_m^M \left[(M-\lambda)^{k+1} + (-1)^k (\lambda - m)^{k+1}\right] f^{(k)}\left(\lambda\right) dE_\lambda
$$
$$
+ T_n\left(A, m, M\right).
$$

Since, by the spectral representation theorem we have

$$
\int_{m-0}^{M} f\left(\lambda\right) dE_\lambda = f\left(A\right), \int_{m-0}^{M} dE_\lambda = 1_H
$$

and

$$
\int_{m-0}^{M} \left[(M-\lambda)^{k+1} + (-1)^k (\lambda - m)^{k+1}\right] f^{(k)}\left(\lambda\right) dE_\lambda
$$
$$
= \left[(M1_H - A)^{k+1} + (-1)^k (A - m1_H)^{k+1}\right] f^{(k)}\left(A\right),
$$

then by (4.176) we deduce the representation (4.172).

Now, if the n-th derivative $f^{(n)}$ is absolutely continuous on $[m, M]$, then

$$\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right) = \int_m^\lambda (t-m)^{n+1} f^{(n+1)}(t)\, dt$$

and

$$\int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t)\right) = \int_\lambda^M (t-M)^{n+1} f^{(n+1)}(t)\, dt$$

where the integrals in the right hand side are taken in the Lebesgue sense.

Utilising the integration by parts formula for the Riemann-Stieltjes integral and the differentiation rule for the Stieltjes integral we have successively

$$\int_{m-0}^M \left(\int_m^\lambda (t-m)^{n+1} f^{(n+1)}(t)\, dt\right) dE_\lambda$$

$$= \left(\int_m^\lambda (t-m)^{n+1} f^{(n+1)}(t)\, dt\right) E_\lambda \Big|_{m-0}^M$$

$$- \int_{m-0}^M (\lambda-m)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda$$

$$= \left(\int_m^M (t-m)^{n+1} f^{(n+1)}(t)\, dt\right) 1_H$$

$$- \int_{m-0}^M (\lambda-m)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda$$

$$= \int_{m-0}^M (\lambda-m)^{n+1} f^{(n+1)}(\lambda)\, (1_H - E_\lambda)\, d\lambda$$

and

$$\int_{m-0}^M \left(\int_\lambda^M (t-M)^{n+1} f^{(n+1)}(t)\, dt\right) dE_\lambda$$

$$= \left(\int_\lambda^M (t-M)^{n+1} f^{(n+1)}(t)\, dt\right) E_\lambda \Big|_{m-0}^M$$

$$+ \int_{m-0}^M (\lambda-M)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda$$

$$= \int_{m-0}^M (\lambda-M)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda$$

and the representation (4.174) is thus obtained. ∎

Remark 319 *Let A be a positive selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some positive real numbers $0 < m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family. Then, for $n \geq 1$, we have the equality*

$$\ln A = [\ln I(m, M)] 1_H + \frac{1}{M - m} \tag{4.177}$$

$$\times \sum_{k=1}^{n} \frac{1}{k(k+1)} \left[(A - m1_H)^{k+1} + (-1)^k (M1_H - A)^{k+1} \right] A^{-k}$$

$$+ \frac{1}{(M - m)(n + 1)}$$

$$\times \left[\int_{m-0}^{M} \left[(\lambda - m)^{n+1} (1_H - E_\lambda) + (\lambda - M)^{n+1} E_\lambda \right] \lambda^{-n-1} d\lambda \right],$$

where $I(m, M)$ is the identric mean and is defined by

$$I(m, M) = \begin{cases} \frac{1}{e} \left(\frac{M^M}{m^m} \right)^{1/(M-m)} & \text{if } M \neq m; \\ \\ M & \text{if } M = m. \end{cases}$$

Remark 320 *If we introduce the exponential mean by*

$$E(m, M) = \begin{cases} \frac{\exp M - \exp m}{M - m} & \text{if } M \neq m; \\ \\ M & \text{if } M = m \end{cases}$$

and applying the identity (4.172) for the exponential function, we have

$$\left[1_H + \frac{1}{M - m} \sum_{k=1}^{n} \frac{1}{(k+1)!} \left[(M1_H - A)^{k+1} + (-1)^k (A - m1_H)^{k+1} \right] \right]$$

$$\times \exp A - E(m, M) 1_H \tag{4.178}$$

$$= \frac{(-1)^n}{(M - m)(n + 1)!} \int_{m-0}^{M} \left[(\lambda - m)^{n+1} (1_H - E_\lambda) + (\lambda - M)^{n+1} E_\lambda \right] e^\lambda d\lambda$$

where A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ is its spectral family.

4.9.2   Error Bounds for $f^{(n)}$ of Bounded Variation

From the identity (4.172), we define for any $x, y \in H$

$$T_n\left(A, m, M; x, y\right) \tag{4.179}$$

$$:= \langle f\left(A\right) x, y \rangle + \frac{1}{M - m} \sum_{k=1}^{n} \frac{1}{(k+1)!}$$

$$\times \left[\left\langle \left(M1_H - A\right)^{k+1} f^{(k)}\left(A\right) x, y \right\rangle + (-1)^k \left\langle \left(A - m1_H\right)^{k+1} f^{(k)}\left(A\right) x, y \right\rangle \right]$$

$$- \left(\frac{1}{M - m} \int_m^M f\left(t\right) dt \right) \langle x, y \rangle.$$

We have the following result concerning bounds for the absolute value of $T_n\left(A, m, M; x, y\right)$ when the n-th derivative $f^{(n)}$ is of bounded variation:

Theorem 321 (Dragomir, 2010, [8]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\left(A\right) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ and let n be an integer with $n \geq 1$.*

1. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then we have the inequalities

$$\left| T_n\left(A, m, M; x, y\right) \right| \tag{4.180}$$

$$\leq \frac{1}{(M - m)(n + 1)!} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\times \max_{\lambda \in [m, M]} \left[(\lambda - m)^{n+1} \bigvee_m^\lambda \left(f^{(n)} \right) + (M - \lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)} \right) \right]$$

$$\leq \frac{(M - m)^n}{(n + 1)!} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \bigvee_m^M \left(f^{(n)} \right) \leq \frac{(M - m)^n}{(n + 1)!} \bigvee_m^M \left(f^{(n)} \right) \|x\| \|y\|$$

for any $x, y \in H$.

2. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on the interval $[m, M]$, then we have the inequalities

$$\left| T_n\left(A, m, M; x, y\right) \right| \leq \frac{L_n\left(M - m\right)^{n+1}}{(n + 2)!} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \tag{4.181}$$

$$\leq \frac{L_n\left(M - m\right)^{n+1}}{(n + 2)!} \|x\| \|y\|$$

for any $x, y \in H$.

3. If $f : I \to \mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is monotonic nondecreasing on the interval $[m, M]$, then we have the inequalities

$$|T_n(A, m, M; x, y)| \tag{4.182}$$

$$\leq \frac{1}{(M-m)(n+1)!} \bigvee_m^M (\langle E_{(\cdot)}x, y \rangle)$$

$$\times \max_{\lambda \in [m,M]} \left[f^{(n)}(\lambda) \left((\lambda - m)^{n+1} - (M - \lambda)^{n+1} \right) \right.$$

$$+ (n+1) \left[\int_\lambda^M (M-t)^n f^{(n)}(t) \, dt - \int_m^\lambda (t-m)^n f^{(n)}(t) \, dt \right] \right]$$

$$\leq \frac{1}{(M-m)(n+1)!} \max_{\lambda \in [m,M]} \left[(\lambda - m)^{n+1} \left[f^{(n)}(\lambda) - f^{(n)}(m) \right] \right.$$

$$\left. + (M-\lambda)^{n+1} \left[f^{(n)}(M) - f^{(n)}(\lambda) \right] \right] \bigvee_m^M (\langle E_{(\cdot)}x, y \rangle)$$

$$\leq \frac{(M-m)^n}{(n+1)!} \bigvee_m^M (\langle E_{(\cdot)}x, y \rangle) \left[f^{(n)}(M) - f^{(n)}(m) \right]$$

$$\leq \frac{(M-m)^n}{(n+1)!} \left[f^{(n)}(M) - f^{(n)}(m) \right] \|x\| \|y\|$$

for any $x, y \in H$.

Proof. 1. By the identity (4.173) we have for any $x, y \in H$ that

$$T_n(A, m, M; x, y) := \frac{(-1)^n}{(M-m)(n+1)!} \tag{4.183}$$

$$\times \left[\int_{m-0}^M \left(\int_m^\lambda (t-m)^{n+1} \, d\left(f^{(n)}(t) \right) \right) d\langle E_\lambda x, y \rangle \right.$$

$$\left. + \int_{m-0}^M \left(\int_\lambda^M (t-M)^{n+1} \, d\left(f^{(n)}(t) \right) \right) d\langle E_\lambda x, y \rangle \right].$$

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v), \tag{4.184}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Taking the modulus in (4.183) and utilizing the property (4.184), we have successively that

$$|T_n\left(A, m, M; x, y\right)| \tag{4.185}$$

$$= \frac{1}{(M-m)(n+1)!}$$

$$\times \left| \int_{m-0}^{M} \left[\int_{m}^{\lambda} (t-m)^{n+1}\, d\left(f^{(n)}(t)\right) \right. \right.$$

$$\left. \left. + \int_{\lambda}^{M} (t-M)^{n+1}\, d\left(f^{(n)}(t)\right) \right] d\left\langle E_\lambda x, y\right\rangle \right|$$

$$\leq \frac{1}{(M-m)(n+1)!} \bigvee_{m}^{M} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)$$

$$\times \max_{\lambda \in [m,M]} \left| \int_{m}^{\lambda} (t-m)^{n+1}\, d\left(f^{(n)}(t)\right) + \int_{\lambda}^{M} (t-M)^{n+1}\, d\left(f^{(n)}(t)\right) \right|$$

for any $x, y \in H$.

By the same property (4.184) we have for $\lambda \in (m, M)$ that

$$\left| \int_{m}^{\lambda} (t-m)^{n+1}\, d\left(f^{(n)}(t)\right) \right| \leq \max_{t \in [m,\lambda]} (t-m)^{n+1} \bigvee_{m}^{\lambda} \left(f^{(n)}\right)$$

$$= (\lambda-m)^{n+1} \bigvee_{m}^{\lambda} \left(f^{(n)}\right)$$

and

$$\left| \int_{\lambda}^{M} (t-M)^{n+1}\, d\left(f^{(n)}(t)\right) \right| \leq \max_{t \in [\lambda,M]} (M-t)^{n+1} \bigvee_{\lambda}^{M} \left(f^{(n)}\right)$$

$$= (M-\lambda)^{n+1} \bigvee_{\lambda}^{M} \left(f^{(n)}\right)$$

which produce the inequality

$$\left| \int_{m}^{\lambda} (t-m)^{n+1}\, d\left(f^{(n)}(t)\right) + \int_{\lambda}^{M} (t-M)^{n+1}\, d\left(f^{(n)}(t)\right) \right| \tag{4.186}$$

$$\leq (\lambda-m)^{n+1} \bigvee_{m}^{\lambda} \left(f^{(n)}\right) + (M-\lambda)^{n+1} \bigvee_{\lambda}^{M} \left(f^{(n)}\right).$$

Taking the maximum over $\lambda \in [m, M]$ in (4.186) and utilizing (4.185) we deduce the first inequality in (4.180).

Now observe that

$$(\lambda - m)^{n+1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M - \lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)}\right)$$

$$\leq \max\left\{(\lambda - m)^{n+1}, (M - \lambda)^{n+1}\right\} \left[\bigvee_m^\lambda \left(f^{(n)}\right) + \bigvee_\lambda^M \left(f^{(n)}\right)\right]$$

$$= \max\left\{(\lambda - m)^{n+1}, (M - \lambda)^{n+1}\right\} \bigvee_m^M \left(f^{(n)}\right)$$

$$= \left[\frac{1}{2}(M - m) + \left|\lambda - \frac{m + M}{2}\right|\right]^{n+1} \bigvee_m^M \left(f^{(n)}\right)$$

giving that

$$\max_{\lambda \in [m, M]} \left[(\lambda - m)^{n+1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M - \lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)}\right)\right]$$

$$\leq (M - m)^{n+1} \bigvee_m^M \left(f^{(n)}\right)$$

and the second inequality in (4.180) is proved.

The last part of (4.180) follows by the Total Variation Schwarz's inequality and we omit the details.

2. Now, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t) \, dv(t)\right| \leq L \int_a^b |p(t)| \, dt. \tag{4.187}$$

By the property (4.187) we have for $\lambda \in (m, M)$ that

$$\left|\int_m^\lambda (t - m)^{n+1} \, d\left(f^{(n)}(t)\right)\right| \leq L_n \int_m^\lambda (t - m)^{n+1} \, d(t) = \frac{L_n}{n+2}(\lambda - m)^{n+2}$$

and

$$\left|\int_\lambda^M (t - M)^{n+1} \, d\left(f^{(n)}(t)\right)\right| \leq L_n \int_\lambda^M (M - t)^{n+1} \, dt = \frac{L_n}{n+2}(M - \lambda)^{n+2}.$$

By the inequality (4.185) we then have

$$|T_n (A, m, M; x, y)| \tag{4.188}$$

$$\leq \frac{1}{(M - m)(n + 1)!} \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle)$$

$$\times \max_{\lambda \in [m, M]} \left[\frac{L_n}{n + 2} (\lambda - m)^{n+2} + \frac{L_n}{n + 2} (M - \lambda)^{n+2} \right]$$

$$= \frac{L_n (M - m)^{n+1}}{(n + 2)!} \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \leq \frac{L_n (M - m)^{n+1}}{(n + 2)!} \|x\| \|y\|$$

for any $x, y \in H$ and the inequality (4.181) is proved.

3. Further, from the theory of Riemann-Stieltjes integral it is also well known that if $p : [a, b] \to \mathbb{C}$ is continuous and $v : [a, b] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t) \, dv(t)$ and $\int_a^b |p(t)| \, dv(t)$ exist and

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \int_a^b |p(t)| \, dv(t) \leq \max_{t \in [a, b]} |p(t)| [v(b) - v(a)]. \tag{4.189}$$

On making use of (4.189) we have

$$\left| \int_m^\lambda (t - m)^{n+1} \, d\left(f^{(n)} (t) \right) \right| \leq \int_m^\lambda (t - m)^{n+1} \, d\left(f^{(n)} (t) \right) \tag{4.190}$$

$$\leq (\lambda - m)^{n+1} \left[f^{(n)} (\lambda) - f^{(n)} (m) \right]$$

and

$$\left| \int_\lambda^M (t - M)^{n+1} \, d\left(f^{(n)} (t) \right) \right| \leq \int_\lambda^M (M - t)^{n+1} \, d\left(f^{(n)} (t) \right) \tag{4.191}$$

$$\leq (M - \lambda)^{n+1} \left[f^{(n)} (M) - f^{(n)} (\lambda) \right]$$

for any $\lambda \in (m, M)$.

Integrating by parts in the Riemann-Stieltjes integral, we also have

$$\int_m^\lambda (t - m)^{n+1} \, d\left(f^{(n)} (t) \right)$$

$$= (\lambda - m)^{n+1} f^{(n)} (\lambda) - (n + 1) \int_m^\lambda (t - m)^n f^{(n)} (t) \, dt$$

and

$$\int_\lambda^M (M - t)^{n+1} \, d\left(f^{(n)} (t) \right)$$

$$= (n + 1) \int_\lambda^M (M - t)^n f^{(n)} (t) \, dt - (M - \lambda)^{n+1} f^{(n)} (\lambda)$$

for any $\lambda \in (m, M)$.

Therefore, by adding (4.190) with (4.191) we get

$$
\left| \int_m^\lambda (t - m)^{n+1} \, d\left(f^{(n)}(t) \right) \right| + \left| \int_\lambda^M (t - M)^{n+1} \, d\left(f^{(n)}(t) \right) \right|
$$
$$
\leq \left[f^{(n)}(\lambda) \left((\lambda - m)^{n+1} - (M - \lambda)^{n+1} \right) \right]
$$
$$
+ (n+1) \left[\int_\lambda^M (M - t)^n \, f^{(n)}(t) \, dt - \int_m^\lambda (t - m)^n \, f^{(n)}(t) \, dt \right]
$$
$$
\leq (\lambda - m)^{n+1} \left[f^{(n)}(\lambda) - f^{(n)}(m) \right] + (M - \lambda)^{n+1} \left[f^{(n)}(M) - f^{(n)}(\lambda) \right]
$$

for any $\lambda \in (m, M)$.

Now, on making use of the inequality (4.185) we deduce (4.182). ∎

Remark 322 *If we use the inequality (4.180) for the function* $\ln$*, then we get the inequality*

$$
|L_n (A, m, M; x, y)| \tag{4.192}
$$
$$
\leq \frac{1}{(M - m) \, n \, (n+1)} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)
$$
$$
\times \max_{\lambda \in [m, M]} \left[(\lambda - m)^{n+1} \frac{\lambda^n - m^n}{\lambda^n m^n} + (M - \lambda)^{n+1} \frac{M^n - \lambda^n}{M^n \lambda^n} \right]
$$
$$
\leq \frac{(M - m)^n \, (M^n - m^n)}{n \, (n+1) \, M^n m^n} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)
$$
$$
\leq \frac{(M - m)^n \, (M^n - m^n)}{n \, (n+1) \, M^n m^n} \, \|x\| \, \|y\|
$$

for any $x, y \in H$*, where*

$$
L_n (A, m, M; x, y) \tag{4.193}
$$
$$
:= \langle \ln Ax, y \rangle - [\ln I (m, M)] \langle x, y \rangle
$$
$$
- \frac{1}{M - m} \sum_{k=1}^n \frac{1}{k \, (k+1)}
$$
$$
\times \left[\left\langle (A - m1_H)^{k+1} A^{-k} x, y \right\rangle + (-1)^k \left\langle (M1_H - A)^{k+1} A^{-k} x, y \right\rangle \right].
$$

If we use the inequality (4.181) for the function ln we get the following bound as well

$$|L_n(A, m, M; x, y)| \qquad (4.194)$$

$$\leq \frac{1}{(n+1)(n+2)} \left(\frac{M}{m} - 1\right)^{n+1} \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle)$$

$$\leq \frac{1}{(n+1)(n+2)} \left(\frac{M}{m} - 1\right)^{n+1} \|x\| \|y\|$$

for any $x, y \in H$.

Remark 323 If we define

$$E_n(A, m, M; x, y) \qquad (4.195)$$

$$:= \left\langle \left[1_H + \frac{1}{M-m} \right.\right.$$

$$\times \sum_{k=1}^{n} \frac{1}{(k+1)!} \left[(M 1_H - A)^{k+1} + (-1)^k (A - m 1_H)^{k+1} \right] \exp A \right] x, y \right\rangle$$

$$- E(m, M) \langle x, y \rangle,$$

then by the inequality (4.180) we have

$$|E_n(A, m, M; x, y)| \qquad (4.196)$$

$$\leq \frac{1}{(M-m)(n+1)!} \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle)$$

$$\times \max_{\lambda \in [m,M]} \left[(\lambda - m)^{n+1} (e^\lambda - e^m) + (M - \lambda)^{n+1} (e^M - e^\lambda) \right]$$

$$\leq \frac{(M-m)^n}{(n+1)!} \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) (e^M - e^m)$$

$$\leq \frac{(M-m)^n}{(n+1)!} (e^M - e^m) \|x\| \|y\|$$

for any $x, y \in H$.

If we use the inequality (4.181) for the function exp we get the following bound as well

$$|E_n(A, m, M; x, y)| \leq \frac{e^M (M-m)^{n+1}}{(n+2)!} \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \qquad (4.197)$$

$$\leq \frac{e^M (M-m)^{n+1}}{(n+2)!} \|x\| \|y\|$$

for any $x, y \in H$.

4.9.3   Error Bounds for $f^{(n)}$ Absolutely Continuous

We consider the Lebesgue norms defined by

$$\|g\|_{[a,b],\infty} := ess \sup_{t\in[a,b]} |g(t)| \text{ if } g \in L_\infty [a,b]$$

and

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p \, dt \right)^{1/p} \text{ if } g \in L_p [a,b], p \geq 1.$$

Theorem 324 (Dragomir, 2010, [8]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ and let n be an integer with $n \geq 1$. If the n-th derivative $f^{(n)}$ is absolutely continuous on $[m, M]$, then*

$$|T_n (A, m, M; x, y)| \tag{4.198}$$

$$\leq \frac{1}{(M-m)(n+1)!}$$

$$\times \int_{m-0}^M \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right| \left| f^{(n+1)} (\lambda) \right| d\lambda.$$

$$\leq \frac{1}{(M-m)(n+1)!}$$

$$\times \begin{cases} B_{n,1} (A, m, M; x, y) \left\| f^{(n)} \right\|_{[m,M],\infty} & \text{if } f^{(n)} \in L_\infty [m, M], \\[2mm] B_{n,p} (A, m, M; x, y) \left\| f^{(n)} \right\|_{[m,M],q} & \text{if } f^{(n)} \in L_q [m, M], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\[2mm] B_{n,\infty} (A, m, M; x, y) \left\| f^{(n)} \right\|_{[m,M],1}, \end{cases}$$

for any $x, y \in H$, where

$$B_{n,p} (A, m, M; x, y)$$

$$:= \left(\int_{m-0}^M \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right|^p d\lambda \right)^{1/p}, p \geq 1$$

and

$$B_{n,\infty} (A, m, M; x, y)$$

$$:= \sup_{t\in[m,M]} \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right|.$$

Proof. Follows from the representation

$$T_n \left(A, m, M; x, y\right)$$
$$= \frac{(-1)^n}{(M - m)(n + 1)!}$$
$$\times \int_{m-0}^{M} \left[(\lambda - m)^{n+1} \left\langle (1_H - E_\lambda) x, y \right\rangle + (\lambda - M)^{n+1} \left\langle E_\lambda x, y \right\rangle \right] f^{(n+1)} (\lambda) \, d\lambda$$

for any $x, y \in H$, by taking the modulus and utilizing the Hölder integral inequality.

The details are omitted. ■

The bounds provided by $B_{n,p} \left(A, m, M; x, y\right)$ are not useful for applications, therefore we will establish in the following some simpler, however coarser bounds.

Proposition 325 (Dragomir, 2010, [8]) *With the above notations, we have*

$$B_{n,\infty} \left(A, m, M; x, y\right) \leq (M - m)^{n+1} \|x\| \|y\|, \qquad (4.199)$$

$$B_{n,1} \left(A, m, M; x, y\right) \leq \frac{\left(2^{n+2} - 1\right)}{(n + 2) \, 2^{n+1}} (M - m)^{n+2} \|x\| \|y\| \qquad (4.200)$$

and for $p > 1$

$$B_{n,p} \left(A, m, M; x, y\right) \qquad (4.201)$$
$$\leq \frac{\left(2^{(n+1)p+1} - 1\right)^{1/p}}{2^{n+1} \left[(n + 1) \, p + 1\right]^{1/p}} (M - m)^{n+1+1/p} \|x\| \|y\|$$

for any $x, y \in H$.

Proof. Utilising the triangle inequality for the modulus we have

$$\left| (\lambda - m)^{n+1} \left\langle (1_H - E_\lambda) x, y \right\rangle + (\lambda - M)^{n+1} \left\langle E_\lambda x, y \right\rangle \right| \qquad (4.202)$$
$$\leq (\lambda - m)^{n+1} \left| \left\langle (1_H - E_\lambda) x, y \right\rangle \right| + (M - \lambda)^{n+1} \left| \left\langle E_\lambda x, y \right\rangle \right|$$
$$\leq \max \left\{ (\lambda - m)^{n+1}, (M - \lambda)^{n+1} \right\} \left[\left| \left\langle (1_H - E_\lambda) x, y \right\rangle \right| + \left| \left\langle E_\lambda x, y \right\rangle \right| \right]$$

for any $x, y \in H$.

Utilising the generalization of Schwarz's inequality for nonnegative self-adjoint operators we have

$$\left| \left\langle (1_H - E_\lambda) x, y \right\rangle \right| \leq \left\langle (1_H - E_\lambda) x, x \right\rangle^{1/2} \left\langle (1_H - E_\lambda) y, y \right\rangle^{1/2}$$

and

$$\left| \left\langle E_\lambda x, y \right\rangle \right| \leq \left\langle E_\lambda x, x \right\rangle^{1/2} \left\langle E_\lambda y, y \right\rangle^{1/2}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Further, by making use of the elementary inequality

$$ac + bd \le \left(a^2 + b^2\right)^{1/2} \left(c^2 + d^2\right)^{1/2}, a, b, c, d \ge 0$$

we have

$$|\langle (1_H - E_\lambda) x, y \rangle| + |\langle E_\lambda x, y \rangle| \quad (4.203)$$
$$\le \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} + \langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2}$$
$$\le \left(\langle (1_H - E_\lambda) x, x \rangle + \langle E_\lambda x, x \rangle \right)^{1/2} \left(\langle (1_H - E_\lambda) y, y \rangle + \langle E_\lambda y, y \rangle \right)^{1/2}$$
$$= \|x\| \, \|y\|$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Combining (4.202) with (4.203) we deduce that

$$\left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right| \quad (4.204)$$
$$\le \max \left\{ (\lambda - m)^{n+1}, (M - \lambda)^{n+1} \right\} \|x\| \, \|y\|$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Taking the supremum over $\lambda \in [m, M]$ in (4.204) we deduce the inequality (4.199).

Now, if we take the power $r \ge 1$ in (4.204) and integrate, then we get

$$\int_{m-0}^{M} \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right|^r d\lambda \quad (4.205)$$
$$\le \|x\|^r \, \|y\|^r \int_{m}^{M} \max \left\{ (\lambda - m)^{(n+1)r}, (M - \lambda)^{(n+1)r} \right\} d\lambda$$
$$= \|x\|^r \, \|y\|^r \left[\int_{m}^{\frac{M+m}{2}} (M - \lambda)^{(n+1)r} d\lambda + \int_{\frac{M+m}{2}}^{M} (\lambda - m)^{(n+1)r} d\lambda \right]$$
$$= \frac{\left(2^{(n+1)r+1} - 1 \right)}{[(n+1)r+1] \, 2^{(n+1)r}} (M - m)^{(n+1)r+1} \|x\|^r \, \|y\|^r$$

for any $x, y \in H$.

Utilizing (4.205) for $r = 1$ we deduce the bound (4.200). Also, by making $r = p$ and then taking the power $1/p$, we deduce the last inequality (4.201).
∎

The following result provides refinements of the inequalities in Proposition 325:

Proposition 326 (Dragomir, 2010, [8]) *With the above notations, we have*

$$B_{n,\infty}(A, m, M; x, y) \tag{4.206}$$
$$\leq \|y\|$$
$$\times \max_{\lambda \in [m,M]} \left[(\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x \rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x \rangle \right]^{1/2}$$
$$\leq (M - m)^{n+1} \|x\| \|y\|,$$

$$B_{n,1}(A, m, M; x, y) \tag{4.207}$$
$$\leq \|y\|$$
$$\times \int_{m-0}^{M} \left[(\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x \rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x \rangle \right]^{1/2} d\lambda$$
$$\leq \frac{(2^{n+2} - 1)}{(n+2) 2^{n+1}} (M - m)^{n+2} \|x\| \|y\|$$

and for $p > 1$

$$B_{n,p}(A, m, M; x, y) \tag{4.208}$$
$$\leq \|y\|$$
$$\times \left(\int_{m-0}^{M} \left[(\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x \rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x \rangle \right]^{p/2} d\lambda \right)^{1/p}$$
$$\leq \frac{(2^{(n+1)p+1} - 1)^{1/p}}{2^{n+1} [(n+1) p + 1]^{1/p}} (M - m)^{n+1+1/p} \|x\| \|y\|$$

for any $x, y \in H$.

Proof. Utilising the Schwarz inequality in H, we have

$$\left| \left\langle (\lambda - m)^{n+1} (1_H - E_\lambda) x + (\lambda - M)^{n+1} E_\lambda x, y \right\rangle \right| \tag{4.209}$$
$$\leq \|y\| \left\| (\lambda - m)^{n+1} (1_H - E_\lambda) x + (\lambda - M)^{n+1} E_\lambda x \right\|$$

for any $x, y \in H$.

Since E_λ are projectors for each $\lambda \in [m, M]$, then we have

$$\left\| (\lambda - m)^{n+1} (1_H - E_\lambda) x + (\lambda - M)^{n+1} E_\lambda x \right\|^2 \qquad (4.210)$$

$$= (\lambda - m)^{2(n+1)} \left\| (1_H - E_\lambda) x \right\|^2$$

$$+ 2 (\lambda - m)^{n+1} (\lambda - M)^{n+1} \operatorname{Re} \langle (1_H - E_\lambda) x, E_\lambda x \rangle$$

$$+ (M - \lambda)^{2(n+1)} \left\| E_\lambda x \right\|^2$$

$$= (\lambda - m)^{2(n+1)} \left\| (1_H - E_\lambda) x \right\|^2 + (M - \lambda)^{2(n+1)} \left\| E_\lambda x \right\|^2$$

$$= (\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x \rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x \rangle$$

$$\leq \|x\|^2 \max \left\{ (\lambda - m)^{2(n+1)}, (M - \lambda)^{2(n+1)} \right\}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

On making use of (4.209) and (4.210) we obtain the following refinement of (4.204)

$$\left| \left\langle (\lambda - m)^{n+1} (1_H - E_\lambda) x + (\lambda - M)^{n+1} E_\lambda x, y \right\rangle \right| \qquad (4.211)$$

$$\leq \|y\| \left[(\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x \rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x \rangle \right]^{1/2}$$

$$\leq \max \left\{ (\lambda - m)^{n+1}, (M - \lambda)^{n+1} \right\} \|x\| \|y\|$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

The proof now follows the lines of the proof from Proposition 325 and we omit the details. ∎

Remark 327 *One can apply Theorem 324 and Proposition 325 for particular functions including the exponential and logarithmic function. However the details are left to the interested reader.*

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5
Inequalities of Trapezoidal Type

5.1 Introduction

From a complementary viewpoint to Ostrowski/mid-point inequalities, trapezoidal type inequality provide a priory error bounds in approximating the Riemann integral by a (generalized) trapezoidal formula.

Just like in the case of Ostrowski's inequality the development of these kind of results have registered a sharp growth in the last decade with more than 50 papers published, as one can easily asses this by performing a search with the key word "trapezoid" and "inequality" in the title of the papers reviewed by MathSciNet data base of the American Mathematical Society.

Numerous extensions, generalisations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Probability Theory and other fields have been also given.

In the present chapter we present some recent results obtained by the author in extending trapezoidal type inequality in various directions for continuous functions of selfadjoint operators in complex Hilbert spaces. As far as we know, the obtained results are new with no previous similar results ever obtained in the literature.

Applications for some elementary functions of operators such as the power function, the logarithmic and exponential functions are provided as well.

5.2   Scalar Trapezoidal Type Inequalities

In Classical Analysis a *trapezoidal type inequality* is an inequality that provides upper and/or lower bounds for the quantity

$$\frac{f(a) + f(b)}{2}(b - a) - \int_a^b f(t)\, dt,$$

that is the error in approximating the integral by a trapezoidal rule, for various classes of integrable functions f defined on the compact interval $[a, b]$.

In order to introduce the reader to some of the well known results and prepare the background for considering a similar problem for functions of selfadjoint operators in Hilbert spaces, we mention the following inequalities.

The case of functions of bounded variation was obtained in [2] (see also [1, p. 68]):

Theorem 328 *Let $f : [a, b] \to \mathbb{C}$ be a function of bounded variation. We have the inequality*

$$\left| \int_a^b f(t)\, dt - \frac{f(a) + f(b)}{2}(b - a) \right| \leq \frac{1}{2}(b - a) \bigvee_a^b (f), \qquad (5.1)$$

where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a, b]$. The constant $\frac{1}{2}$ is the best possible one.

This result may be improved if one assumes the monotonicity of f as follows (see [1, p. 76]):

Theorem 329 *Let $f : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$. Then we have the inequalities:*

$$\left| \int_a^b f(t)\, dt - \frac{f(a) + f(b)}{2}(b - a) \right| \qquad (5.2)$$

$$\leq \frac{1}{2}(b - a)[f(b) - f(a)] - \int_a^b sgn\left(t - \frac{a + b}{2}\right) f(t)\, dt$$

$$\leq \frac{1}{2}(b - a)[f(b) - f(a)].$$

The above inequalities are sharp.

If the mapping is Lipschitzian, then the following result holds as well [3] (see also [1, p. 82]).

Theorem 330 *Let $f : [a, b] \to \mathbb{C}$ be an $L-$Lipschitzian function on $[a, b]$, i.e., f satisfies the condition:*

$$|f(s) - f(t)| \le L|s - t| \quad \text{for any} \ \ s, t \in [a, b] \quad (L > 0 \ \text{is given}). \quad \text{(L)}$$

Then we have the inequality:

$$\left| \int_a^b f(t) \, dt - \frac{f(a) + f(b)}{2} (b - a) \right| \le \frac{1}{4} (b - a)^2 L. \quad (5.3)$$

The constant $\frac{1}{4}$ is best in (5.3).

If we would assume absolute continuity for the function f, then the following estimates in terms of the Lebesgue norms of the derivative f' hold [1, p. 93].

Theorem 331 *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$. Then we have*

$$\left| \int_a^b f(t) \, dt - \frac{f(a) + f(b)}{2} (b - a) \right| \quad (5.4)$$

$$\le \begin{cases} \dfrac{1}{4} (b - a)^2 \, \|f'\|_\infty & \text{if} \ \ f' \in L_\infty [a, b] \, ; \\[2ex] \dfrac{1}{2 (q + 1)^{\frac{1}{q}}} (b - a)^{1 + 1/q} \, \|f'\|_p & \text{if} \ \ f' \in L_p [a, b] \, , \\[1ex] & \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \dfrac{1}{2} (b - a) \, \|f'\|_1 \, , \end{cases}$$

where $\|\cdot\|_p \ (p \in [1, \infty])$ are the Lebesgue norms, i.e.,

$$\|f'\|_\infty = ess \sup_{s \in [a,b]} |f'(s)|$$

and

$$\|f'\|_p := \left(\int_a^b |f'(s)| \, ds \right)^{\frac{1}{p}}, \quad p \ge 1.$$

The case of convex functions is as follows [4]:

Theorem 332 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Then we have the inequalities*

$$\frac{1}{8} (b - a)^2 \left[f'_+ \left(\frac{a + b}{2} \right) - f'_- \left(\frac{a + b}{2} \right) \right] \tag{5.5}$$

$$\leq \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(t) \, dt$$

$$\leq \frac{1}{8} (b - a)^2 \left[f'_- (b) - f'_+ (a) \right].$$

The constant $\frac{1}{8}$ is sharp in both sides of (5.5).

For other scalar trapezoidal type inequalities, see [1].

5.3   Trapezoidal Vector Inequalities

5.3.1   Some General Results

With the notations introduced above, we consider in this paper the problem of bounding the error

$$\frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle$$

in approximating $\langle f(A) x, y \rangle$ by the trapezoidal type formula $\frac{f(M)+f(m)}{2} \cdot \langle x, y \rangle$, where x, y are vectors in the Hilbert space H, f is a continuous functions of the selfadjoint operator A with the spectrum in the compact interval of real numbers $[m, M]$. Applications for some particular elementary functions are also provided. The following result holds:

Theorem 333 (Dragomir, 2010, [5]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \rightarrow \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \tag{5.6}$$

$$\leq \frac{1}{2} \max_{\lambda \in [m, M]} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \right.$$

$$\left. + \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} \right] \bigvee_m^M (f)$$

$$\leq \frac{1}{2} \|x\| \|y\| \bigvee_m^M (f)$$

for any $x, y \in H$.

Proof. If $f, u : [m, M] \to \mathbb{C}$ are such that the Riemann-Stieltjes integral $\int_a^b f(t) \, du(t)$ exists, then a simple integration by parts reveals the identity

$$\int_a^b f(t) \, du(t) = \frac{f(a) + f(b)}{2} \, [u(b) - u(a)] \tag{5.7}$$

$$- \int_a^b \left[u(t) - \frac{u(a) + u(b)}{2} \right] df(t).$$

If we write the identity (5.7) for $u(\lambda) = \langle E_\lambda x, y \rangle$, then we get

$$\int_{m-0}^M f(\lambda) \, d(\langle E_\lambda x, y \rangle) = \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle$$

$$- \int_{m-0}^M \left(\langle E_\lambda x, y \rangle - \frac{1}{2} \langle x, y \rangle \right) df(\lambda)$$

which gives the following identity of interest in itself

$$\frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \tag{5.8}$$

$$= \frac{1}{2} \int_{m-0}^M [\langle E_\lambda x, y \rangle + \langle (E_\lambda - 1_H) x, y \rangle] \, df(\lambda),$$

for any $x, y \in H$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function and $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v) \tag{5.9}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising the property (5.9), we have from (5.8) that

$$\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \tag{5.10}$$

$$\leq \frac{1}{2} \max_{\lambda \in [m,M]} |\langle E_\lambda x, y \rangle + \langle (E_\lambda - 1_H) x, y \rangle| \bigvee_m^M (f)$$

$$\leq \frac{1}{2} \left[\max_{\lambda \in [m,M]} [|\langle E_\lambda x, y \rangle| + |\langle (1_H - E_\lambda) x, y \rangle|] \right] \bigvee_m^M (f).$$

If P is a nonnegative operator on H, i.e., $\langle Px, x \rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in the Hilbert space H

$$|\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle,$$ (5.11)

for any $x, y \in H$.

On applying the inequality (5.11) we have

$$|\langle E_\lambda x, y \rangle| \leq \langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2}$$

and

$$|\langle (1_H - E_\lambda) x, y \rangle| \leq \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2},$$

which, together with the elementary inequality for $a, b, c, d \geq 0$

$$ab + cd \leq \left(a^2 + c^2\right)^{1/2} \left(b^2 + d^2\right)^{1/2}$$

produce the inequalities

$$
\begin{aligned}
&|\langle E_\lambda x, y \rangle| + |\langle (1_H - E_\lambda) x, y \rangle| \qquad\qquad\qquad (5.12)\\
&\leq \langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} + \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2}\\
&\leq \left(\langle E_\lambda x, x \rangle + \langle (1_H - E_\lambda) x, x \rangle\right)\left(\langle E_\lambda y, y \rangle + \langle (1_H - E_\lambda) y, y \rangle\right)\\
&= \|x\| \, \|y\|
\end{aligned}
$$

for any $x, y \in H$.

On utilizing (5.10) and taking the maximum in (5.12) we deduce the desired result (5.6). ∎

The case of Lipschitzian functions may be useful for applications:

Theorem 334 (Dragomir, 2010, [5]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$
\left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \qquad\qquad (5.13)
$$

$$
\leq \frac{1}{2} L \int_{m-0}^{M} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \right.
$$

$$
\left. + \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} \right] d\lambda
$$

$$
\leq \frac{1}{2} (M - m) L \|x\| \, \|y\|
$$

for any $x, y \in H$.

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq L \int_a^b |p(t)| \, dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (5.8) that

$$\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \tag{5.14}$$

$$\leq \frac{1}{2} L \int_{m-0}^{M} |\langle E_\lambda x, y \rangle + \langle (E_\lambda - 1_H) x, y \rangle| \, d\lambda,$$

$$\leq \frac{1}{2} L \int_{m-0}^{M} [|\langle E_\lambda x, y \rangle| + |\langle (1_H - E_\lambda) x, y \rangle|] \, d\lambda,$$

for any $x, y \in H$.

Further, integrating (5.12) on $[m, M]$ we have

$$\int_{m-0}^{M} [|\langle E_\lambda x, y \rangle| + |\langle (1_H - E_\lambda) x, y \rangle|] \, d\lambda \tag{5.15}$$

$$\leq \int_{m-0}^{M} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \right.$$

$$\left. + \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} \right] d\lambda$$

$$\leq (M - m) \|x\| \|y\|$$

which together with (5.14) produces the desired result (5.13). ∎

5.3.2   Other Trapezoidal Vector Inequalities

The following result provides a different perspective in bounding the error in the trapezoidal approximation:

Theorem 335 (Dragomir, 2010, [5]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Assume that $f : [m, M] \to$*

$\mathbb{C}$ *is a continuous function on* $[m, M]$*. Then we have the inequalities*

$$\left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \tag{5.16}$$

$$\leq \begin{cases} \max_{\lambda \in [m,M]} \left| \left\langle E_\lambda x - \frac{1}{2}x, y \right\rangle \right| \bigvee_m^M (f) & \text{if } f \text{ is of bounded variation} \\[2ex] L \int_{m-0}^M \left| \left\langle E_\lambda x - \frac{1}{2}x, y \right\rangle \right| d\lambda & \text{if } f \text{ is } L \text{ Lipschitzian} \\[2ex] \int_{m-0}^M \left| \left\langle E_\lambda x - \frac{1}{2}x, y \right\rangle \right| df(\lambda) & \text{if } f \text{ is nondecreasing} \end{cases}$$

$$\leq \frac{1}{2} \|x\| \|y\| \begin{cases} \bigvee_m^M (f) & \text{if } f \text{ is of bounded variation} \\[2ex] L(M - m) & \text{if } f \text{ is } L \text{ Lipschitzian} \\[2ex] (f(M) - f(m)) & \text{if } f \text{ is nondecreasing} \end{cases}$$

for any $x, y \in H$.

Proof. From (5.10) we have that

$$\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \tag{5.17}$$

$$\leq \frac{1}{2} \max_{\lambda \in [m,M]} \left| \langle E_\lambda x, y \rangle + \langle (E_\lambda - 1_H) x, y \rangle \right| \bigvee_m^M (f)$$

$$= \max_{\lambda \in [m,M]} \left| \left\langle E_\lambda x - \frac{1}{2}x, y \right\rangle \right| \bigvee_m^M (f)$$

for any $x, y \in H$.

Utilising the Schwarz inequality in H and the fact that E_λ are projectors we have successively

$$\left| \left\langle E_\lambda x - \frac{1}{2}x, y \right\rangle \right| \leq \left\| E_\lambda x - \frac{1}{2}x \right\| \|y\| \tag{5.18}$$

$$= \left[\langle E_\lambda x, E_\lambda x \rangle - \langle E_\lambda x, x \rangle + \frac{1}{4} \|x\|^2 \right]^{1/2} \|y\|$$

$$= \frac{1}{2} \|x\| \|y\|$$

for any $x, y \in H$, which proves the first branch in (5.16).

The second inequality follows from (5.14).

From the theory of Riemann-Stieltjes integral is well known that if $p : [a, b] \rightarrow \mathbb{C}$ is of bounded variation and $v : [a, b] \rightarrow \mathbb{R}$ is continuous and

monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\, dv(t)$ and $\int_a^b |p(t)|\, dv(t)$ exist and

$$\left| \int_a^b p(t)\, dv(t) \right| \le \int_a^b |p(t)|\, dv(t).$$
(5.19)

From the representation (5.8) we then have

$$\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right|$$
(5.20)

$$\le \frac{1}{2} \int_{m-0}^M |\langle E_\lambda x, y \rangle + \langle (E_\lambda - 1_H) x, y \rangle|\, df(\lambda)$$

$$= \int_{m-0}^M \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right|\, df(\lambda)$$

for any $x, y \in H$, from which we obtain the last branch in (5.16). ∎

We recall that a function $f : [a, b] \to \mathbb{C}$ is called $r - H$-Hölder continuous with fixed $r \in (0, 1]$ and $H > 0$ if

$$|f(t) - f(s)| \le H|t - s|^r \text{ for any } t, s \in [a, b].$$

We have the following result concerning this class of functions.

Theorem 336 (Dragomir, 2010, [5]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is $r - H$-Hölder continuous on $[m, M]$, then we have the inequality*

$$\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right|$$
(5.21)

$$\le \frac{1}{2^r} H(M - m)^r \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle)$$

$$\le \frac{1}{2^r} H(M - m)^r \|x\| \|y\|$$

for any $x, y \in H$.

Proof. We start with the equality

$$\frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle$$
(5.22)

$$= \int_{m-0}^M \left[\frac{f(M) + f(m)}{2} - f(\lambda) \right] d(\langle E_\lambda x, y \rangle)$$

for any $x, y \in H$, that follows from the spectral representation theorem.

Since the function $\langle E_{(\cdot)}x, y \rangle$ is of bounded variation for any vector $x, y \in H$, by applying the inequality (5.9) we conclude that

$$\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \qquad (5.23)$$

$$\leq \max_{\lambda \in [m,M]} \left| \frac{f(M) + f(m)}{2} - f(\lambda) \right| \bigvee_m^M \left(\langle E_{(\cdot)}x, y \rangle \right)$$

for any $x, y \in H$.

As $f : [m, M] \to \mathbb{C}$ is $r - H$-Hölder continuous on $[m, M]$, then we have

$$\left| \frac{f(M) + f(m)}{2} - f(\lambda) \right| \leq \frac{1}{2} |f(M) - f(\lambda)| + \frac{1}{2} |f(\lambda) - f(m)| \quad (5.24)$$

$$\leq \frac{1}{2} H \left[(M - \lambda)^r + (\lambda - m)^r \right]$$

for any $\lambda \in [m, M]$.

Since, obviously, the function $g_r(\lambda) := (M - \lambda)^r + (\lambda - m)^r, r \in (0, 1)$ has the property that

$$\max_{\lambda \in [m,M]} g_r(\lambda) = g_r \left(\frac{m + M}{2} \right) = 2^{1-r} (M - m)^r,$$

then by (5.23) we deduce the first part of (5.21).

The last part follows by the Total Variation Schwarz's inequality and we omit the details. ∎

5.3.3   Applications for Some Particular Functions

It is obvious that the results established above can be applied for various particular functions of selfadjoint operators. We restrict ourselves here to only two examples, namely the logarithm and the power functions.

1. If we consider the logarithmic function $f : (0, \infty) \to \mathbb{R}, f(t) = \ln t$, then we can state the following result:

Proposition 337 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have*

$$\left| \langle x, y \rangle \ln \sqrt{mM} - \langle \ln Ax, y \rangle \right| \qquad (5.25)$$

$$\leq \ln \left(\frac{M}{m} \right) \times \begin{cases} \frac{1}{2} \max_{\lambda \in [m,M]} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \\ \quad + \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} \right] \\ \\ \max_{\lambda \in [m,M]} \left| \langle E_\lambda x - \frac{1}{2} x, y \rangle \right| \end{cases}$$

$$\leq \frac{1}{2} \|x\| \|y\| \ln \left(\frac{M}{m} \right)$$

and

$$\left| \langle x, y \rangle \ln \sqrt{mM} - \langle \ln Ax, y \rangle \right| \qquad (5.26)$$

$$\leq \frac{1}{m} \times \begin{cases} \frac{1}{2} \int_{m-0}^{M} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \right. \\ \quad \left. + \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} \right] d\lambda \\ \\ \int_{m-0}^{M} \left| \langle E_\lambda x - \frac{1}{2} x, y \rangle \right| d\lambda \end{cases}$$

$$\leq \frac{1}{2} \|x\| \|y\| \left(\frac{M}{m} - 1 \right)$$

and

$$\left| \langle x, y \rangle \ln \sqrt{mM} - \langle \ln Ax, y \rangle \right| \leq \int_{m-0}^{M} \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| \lambda^{-1} d\lambda \quad (5.27)$$

$$\leq \frac{1}{2} \|x\| \|y\| \ln \left(\frac{M}{m} \right)$$

respectively.

The proof is obvious from Theorems 333, 334 and 335 applied for the logarithmic function. The details are omitted.

2. Consider now the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^p$ with $p \in (-\infty, 0) \cup (0, \infty)$. In the case when $p \in (0, 1)$, the function is $p - H$-Hölder continuous with $H = 1$ on any subinterval $[m, M]$ of $[0, \infty)$. By making use of Theorem 336 we can state the following result:

Proposition 338 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 \leq m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for $p \in (0, 1)$ we have*

$$\left| \frac{m^p + M^p}{2} \cdot \langle x, y \rangle - \langle A^p x, y \rangle \right| \leq \frac{1}{2^p} (M - m)^p \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \quad (5.28)$$

$$\leq \frac{1}{2^p} (M - m)^p \|x\| \|y\|,$$

for any $x, y \in H$.

The case of powers $p \geq 1$ is embodied in the following:

Proposition 339 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 \leq m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for $p \geq 1$ and for any $x, y \in H$*

we have

$$\left| \frac{m^p + M^p}{2} \cdot \langle x, y \rangle - \langle A^p x, y \rangle \right| \quad (5.29)$$

$$\leq (M^p - m^p) \times \begin{cases} \frac{1}{2} \max_{\lambda \in [m,M]} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \\ + \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} \right] \\ \\ \max_{\lambda \in [m,M]} \left| \langle E_\lambda x - \frac{1}{2} x, y \rangle \right| \end{cases}$$

$$\leq \frac{1}{2} \|x\| \|y\| (M^p - m^p)$$

and

$$\left| \frac{m^p + M^p}{2} \cdot \langle x, y \rangle - \langle A^p x, y \rangle \right| \quad (5.30)$$

$$\leq p M^{p-1} \times \begin{cases} \frac{1}{2} \int_{m-0}^{M} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \\ + \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} \right] d\lambda \\ \\ \int_{m-0}^{M} \left| \langle E_\lambda x - \frac{1}{2} x, y \rangle \right| d\lambda \end{cases}$$

$$\leq \frac{1}{2} p \|x\| \|y\| M^{p-1}$$

and

$$\left| \frac{m^p + M^p}{2} \cdot \langle x, y \rangle - \langle A^p x, y \rangle \right| \leq p \int_{m-0}^{M} \left| \langle E_\lambda x - \frac{1}{2} x, y \rangle \right| \lambda^{p-1} d\lambda \quad (5.31)$$

$$\leq \frac{1}{2} \|x\| \|y\| (M^p - m^p)$$

respectively.

The proof is obvious from Theorems 333, 334 and 335 applied for the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^p$ with $p \geq 1$. The details are omitted.

The case of negative powers is similar. The details are left to the interested reader.

5.4   Generalised Trapezoidal Inequalities

5.4.1   Some Vector Inequalities

In the present section we are interested in providing error bounds for approximating $\langle f(A) x, y \rangle$ with the quantity

$$\frac{1}{M - m} \left[f(m) (M \langle x, y \rangle - \langle A x, y \rangle) + f(M) (\langle A x, y \rangle - m \langle x, y \rangle) \right] \quad (5.32)$$

where $x, y \in H$, which is a generalized trapezoid formula. Applications for some particular functions are provided as well. The following representation is of interest in itself and will be useful in deriving our inequalities later as well:

Lemma 340 (Dragomir, 2010, [6]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\,(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$, then we have the representation*

$$\left\langle \left[\frac{f\,(m)\,(M1_H - A) + f\,(M)\,(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f\,(A)\,x, y \rangle \quad (5.33)$$

$$= \int_{m-0}^{M} \langle E_t x, y \rangle\, df\,(t) - \frac{f\,(M) - f\,(m)}{M - m} \int_{m-0}^{M} \langle E_t x, y \rangle\, dt$$

$$= \int_{m-0}^{M} \left[\langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^{M} \langle E_s x, y \rangle\, ds \right] df\,(t)$$

for any $x, y \in H$.

Proof. Integrating by parts and utilizing the spectral representation theorem we have

$$\int_{m-0}^{M} \langle E_t x, y \rangle\, df\,(t) = f\,(M) \langle x, y \rangle - \int_{m-0}^{M} f\,(t)\, d \langle E_t x, y \rangle$$

$$= f\,(M) \langle x, y \rangle - \langle f\,(A)\,x, y \rangle$$

and

$$\int_{m-0}^{M} \langle E_t x, y \rangle\, dt = M \langle x, y \rangle - \langle Ax, y \rangle$$

for any $x, y \in H$.

Therefore

$$\int_{m-0}^{M} \langle E_t x, y \rangle\, df\,(t) - \frac{f\,(M) - f\,(m)}{M - m} \int_{m-0}^{M} \langle E_t x, y \rangle\, dt$$

$$= f\,(M) \langle x, y \rangle - \langle f\,(A)\,x, y \rangle - \frac{f\,(M) - f\,(m)}{M - m} \left(M \langle x, y \rangle - \langle Ax, y \rangle \right)$$

$$= \frac{1}{M - m} \left[f\,(m)\,(M \langle x, y \rangle - \langle Ax, y \rangle) + f\,(M)\,(\langle Ax, y \rangle - m \langle x, y \rangle) \right]$$

$$- \langle f\,(A)\,x, y \rangle$$

for any $x, y \in H$, which proves the first equality in (5.33).

The second equality is obvious. ∎

The following result provides error bounds in approximating $\langle f\,(A)\,x, y \rangle$ by the generalized trapezoidal rule (5.32):

Theorem 341 (Dragomir, 2010, [6]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

1. If $f : [m, M] \to \mathbb{C}$ is of bounded variation on $[m, M]$, then

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \quad (5.34)$$

$$\leq \sup_{t \in [m,M]} \left[\frac{t - m}{M - m} \bigvee_{m-0}^{t} (\langle E_{(\cdot)}x, y \rangle) + \frac{M - t}{M - m} \bigvee_{t}^{M} (\langle E_{(\cdot)}x, y \rangle) \right] \bigvee_{m}^{M} (f)$$

$$\leq \bigvee_{m-0}^{M} (\langle E_{(\cdot)}x, y \rangle) \bigvee_{m}^{M} (f) \leq \|x\| \|y\| \bigvee_{m}^{M} (f)$$

for any $x, y \in H$.

2. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \quad (5.35)$$

$$\leq L \int_m^M \left[\frac{t - m}{M - m} \bigvee_{m-0}^{t} (\langle E_{(\cdot)}x, y \rangle) + \frac{M - t}{M - m} \bigvee_{t}^{M} (\langle E_{(\cdot)}x, y \rangle) \right] dt$$

$$\leq L(M - m) \bigvee_{m-0}^{M} (\langle E_{(\cdot)}x, y \rangle) \leq L(M - m) \|x\| \|y\|$$

for any $x, y \in H$.

3. If $f : [m, M] \to \mathbb{R}$ is monotonic nondecreasing on $[m, M]$, then

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \quad (5.36)$$

$$\leq \int_m^M \left[\frac{t - m}{M - m} \bigvee_{m-0}^{t} (\langle E_{(\cdot)}x, y \rangle) + \frac{M - t}{M - m} \bigvee_{t}^{M} (\langle E_{(\cdot)}x, y \rangle) \right] df(t)$$

$$\leq \bigvee_{m-0}^{M} (\langle E_{(\cdot)}x, y \rangle) [f(M) - f(m)] \leq \|x\| \|y\| [f(M) - f(m)]$$

for any $x, y \in H$.

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is a bounded function, $v : [a, b] \to \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists, then the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \sup_{t \in [a,b]} |p(t)| \bigvee_{a}^{b} (v), \quad (5.37)$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Applying this property to the equality (5.33), we have

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \quad (5.38)$$

$$\leq \sup_{t \in [m, M]} \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^{M} \langle E_s x, y \rangle \, ds \right| \bigvee_m^M (f)$$

for any $x, y \in H$.

Now, a simple integration by parts in the Riemann-Stieltjes integral reveals the following equality of interest

$$\langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^{M} \langle E_s x, y \rangle \, ds \quad (5.39)$$

$$= \frac{1}{M - m} \left[\int_{m-0}^{t} (s - m) \, d \langle E_s x, y \rangle + \int_{t}^{M} (s - M) \, d \langle E_s x, y \rangle \right]$$

that holds for any $t \in [m, M]$ and for any $x, y \in H$.

Since the function $v(s) := \langle E_s x, y \rangle$ is of bounded variation on $[m, M]$ for any $x, y \in H$, then on applying the inequality (5.37) once more, we get

$$\left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^{M} \langle E_s x, y \rangle \, ds \right| \quad (5.40)$$

$$\leq \frac{1}{M - m} \left[\left| \int_{m-0}^{t} (s - m) \, d \langle E_s x, y \rangle \right| + \left| \int_{t}^{M} (s - M) \, d \langle E_s x, y \rangle \right| \right]$$

$$\leq \frac{t - m}{M - m} \bigvee_{m-0}^{t} (\langle E_{(\cdot)} x, y \rangle) + \frac{M - t}{M - m} \bigvee_{t}^{M} (\langle E_{(\cdot)} x, y \rangle)$$

that holds for any $t \in [m, M]$ and for any $x, y \in H$.

Now, taking the supremum in (5.40) and taking into account that

$$\bigvee_{m-0}^{t} (\langle E_{(\cdot)} x, y \rangle), \bigvee_{t}^{M} (\langle E_{(\cdot)} x, y \rangle) \leq \bigvee_{m-0}^{M} (\langle E_{(\cdot)} x, y \rangle)$$

for any $t \in [m, M]$ and for any $x, y \in H$, we deduce the first and the second inequality in (5.34).

The last part of (5.34) follows by the Total Variation Schwarz's inequality and we omit the details.

Now, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\,dv(t) \right| \le L \int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (5.33) that

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \quad (5.41)$$

$$\le L \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle\,ds \right|\,dt$$

for any $x, y \in H$.

Further on, by utilizing (5.39) we can state that

$$\int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle\,ds \right|\,dt$$

$$\le \frac{1}{M - m} \int_{m-0}^M \left[\left| \int_{m-0}^t (s - m)\,d\langle E_s x, y \rangle \right| + \left| \int_t^M (s - M)\,d\langle E_s x, y \rangle \right| \right]\,dt$$

$$\le \int_{m-0}^M \left[\frac{t - m}{M - m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M - t}{M - m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \right]\,dt$$

$$\le (M - m) \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle)$$

for any $x, y \in H$, which proves the desired result (5.35).

From the theory of Riemann-Stieltjes integral it is also well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left| \int_a^b p(t)\,dv(t) \right| \le \int_a^b |p(t)|\,dv(t).$$

From the representation (5.33) we then have

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \quad (5.42)$$

$$\le \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle\,ds \right|\,df(t)$$

for any $x, y \in H$.

Further on, by utilizing (5.39) we can state that

$$\int_{m-0}^{M} \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^{M} \langle E_s x, y \rangle \, ds \right| df(t)$$

$$\leq \frac{1}{M-m} \int_{m-0}^{M} \left[\left| \int_{m-0}^{t} (s-m) \, d \langle E_s x, y \rangle \right| + \left| \int_{t}^{M} (s-M) \, d \langle E_s x, y \rangle \right| \right] df(t)$$

$$\leq \int_{m-0}^{M} \left[\frac{t-m}{M-m} \bigvee_{m-0}^{t} (\langle E_{(\cdot)} x, y \rangle) + \frac{M-t}{M-m} \bigvee_{t}^{M} (\langle E_{(\cdot)} x, y \rangle) \right] df(t)$$

$$\leq (f(M) - f(m)) \bigvee_{m-0}^{M} (\langle E_{(\cdot)} x, y \rangle)$$

for any $x, y \in H$, which proves the desired result (5.36). ∎

A different approach for Lipschitzian functions is incorporated in:

Theorem 342 (Dragomir, 2010, [6]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then*

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \quad (5.43)$$

$$\leq L \|y\| \int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x \, ds \right\| dt \leq \frac{1}{2} L (M-m) \|x\| \|y\|$$

for any $x, y \in H$.

Proof. We will use the inequality (5.41) for which a different upper bound will be provided.

By the Schwarz inequality in H we have that

$$\int_{m-0}^{M} \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^{M} \langle E_s x, y \rangle \, ds \right| dt \quad (5.44)$$

$$= \int_{m-0}^{M} \left| \left\langle \left[E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x \, ds \right], y \right\rangle \right| dt$$

$$\leq \|y\| \int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x \, ds \right\| dt$$

for any $x, y \in H$.

On utilizing the Cauchy-Buniakovski-Schwarz integral inequality we may state that

$$
\int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\| dt \tag{5.45}
$$

$$
\leq (M-m)^{1/2} \left(\int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\|^2 dt \right)^{1/2}
$$

for any $x \in H$.

Observe that the following equalities of interest hold and they can be easily proved by direct calculations

$$
\frac{1}{M-m} \int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\|^2 dt \tag{5.46}
$$

$$
= \frac{1}{M-m} \int_{m-0}^{M} \| E_t x \|^2 dt - \left\| \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\|^2
$$

and

$$
\frac{1}{M-m} \int_{m-0}^{M} \| E_t x \|^2 dt - \left\| \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\|^2 \tag{5.47}
$$

$$
= \frac{1}{M-m} \int_{m-0}^{M} \left\langle E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds, E_t x - \frac{1}{2} x \right\rangle dt
$$

for any $x \in H$.

By (5.45), (5.46) and (5.47) we get

$$
\int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\| dt \tag{5.48}
$$

$$
\leq (M-m)^{1/2} \left(\int_{m-0}^{M} \left\langle E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds, E_t x - \frac{1}{2} x \right\rangle dt \right)^{1/2}
$$

for any $x \in H$.

On making use of the Schwarz inequality in H we also have

$$\int_{m-0}^{M} \left\langle E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds, E_t x - \frac{1}{2} x \right\rangle dt \qquad (5.49)$$

$$\leq \int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\| \left\| E_t x - \frac{1}{2} x \right\| dt$$

$$= \frac{1}{2} \|x\| \int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\| dt,$$

where we used the fact that E_t are projectors, and in this case we have

$$\left\| E_t x - \frac{1}{2} x \right\|^2 = \|E_t x\|^2 - \langle E_t x, x \rangle + \frac{1}{4} \|x\|^2$$

$$= \langle E_t^2 x, x \rangle - \langle E_t x, x \rangle + \frac{1}{4} \|x\|^2 = \frac{1}{4} \|x\|^2$$

for any $t \in [m, M]$ for any $x \in H$.

From (5.48) and (5.49) we get

$$\int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\| dt \qquad (5.50)$$

$$\leq (M-m)^{1/2} \left(\frac{1}{2} \|x\| \int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\| dt \right)^{1/2}$$

which is clearly equivalent with the following inequality of interest in itself

$$\int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\| dt \leq \frac{1}{2} \|x\| (M-m) \qquad (5.51)$$

for any $x \in H$.

This proves the last part of (5.43). ∎

5.4.2 Applications for Particular Functions

It is obvious that the above results can be applied for various particular functions. However, we will restrict here only to the power and logarithmic functions.

1. Consider now the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^p$ with $p \neq 0$. On applying Theorem 342 we can state the following proposition:

Proposition 343 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 \leq m < M$*

and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities

$$\left| \left\langle \left[\frac{m^p (M1_H - A) + M^p (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle A^p x, y \rangle \right| \quad (5.52)$$

$$\leq B_p \|y\| \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\| dt$$

$$\leq \frac{1}{2} B_p (M - m) \|x\| \|y\|$$

where

$$B_p = p \times \begin{cases} M^{p-1} & \text{if } p \geq 1 \\ \\ m^{p-1} & \text{if } 0 < p < 1, m > 0 \end{cases}$$

and

$$B_p = (-p) m^{p-1} \text{ if } p < 0, m > 0.$$

2. The case of logarithmic function is as follows:

Proposition 344 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\left| \left\langle \left[\frac{(M1_H - A) \ln m + (A - m1_H) \ln M}{M - m} \right] x, y \right\rangle - \langle \ln A x, y \rangle \right| \quad (5.53)$$

$$\leq \frac{1}{m} \|y\| \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\| dt$$

$$\leq \frac{1}{2} \left(\frac{M}{m} - 1 \right) \|x\| \|y\|.$$

5.5   More Generalised Trapezoidal Inequalities

5.5.1   Other Vector Inequalities

The following result for general continuous functions holds:

Theorem 345 (Dragomir, 2010, [7]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is*

continuous on $[m, M]$*, then we have the inequalities:*

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \quad (5.54)$$

$$\leq \left[\max_{t \in [m,M]} f(t) - \min_{t \in [m,M]} f(t) \right] \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \left[\max_{t \in [m,M]} f(t) - \min_{t \in [m,M]} f(t) \right] \|x\| \|y\|$$

for any $x, y \in H.$

Proof. We observe that, by the spectral representation theorem, we have the equality

$$\left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \quad (5.55)$$

$$= \int_{m-0}^{M} \Phi_f(t) \, d(\langle E_t x, y \rangle)$$

for any $x, y \in H$, where $\Phi_f : [m, M] \to \mathbb{R}$ is given by

$$\Phi_f(t) = \frac{1}{M - m} \left[(M - t) f(m) + (t - m) f(M) \right] - f(t).$$

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function and $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \sup_{t \in [a,b]} |p(t)| \bigvee_a^b (v), \quad (5.56)$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Now, if we denote by $\gamma := \min_{t \in [m,M]} f(t)$ and by $\Gamma := \max_{t \in [m,M]} f(t)$ then we have

$$\gamma (M - t) \leq (M - t) f(m) \leq \Gamma (M - t),$$
$$\gamma (t - m) \leq (t - m) f(M) \leq \Gamma (t - m)$$

and

$$-(M - m) \Gamma \leq -(M - m) f(t) \leq -\gamma (M - m)$$

for any $t \in [m, M]$. If we add these three inequalities, then we get

$$-(M - m)(\Gamma - \gamma) \leq (M - m) \Phi_f(t) \leq (M - m)(\Gamma - \gamma)$$

for any $t \in [m, M]$, which shows that

$$|\Phi_f(t)| \leq \Gamma - \gamma \text{ for any } t \in [m, M]. \qquad (5.57)$$

On applying the inequality (5.56) for the representation (5.55) we have from (5.57) that

$$\left| \int_{m-0}^{M} \Phi_f(t) \, d\left(\langle E_t x, y \rangle\right) \right| \leq (\Gamma - \gamma) \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle\right)$$

for any $x, y \in H$, which proves the first part of (5.54).

The last part of (5.54) follows by the Total Variation Schwarz's inequality and we omit the details. ∎

When the generating function is of bounded variation, we have the following result.

Theorem 346 (Dragomir, 2010, [7]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then we have the inequalities:*

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \qquad (5.58)$$

$$\leq \max_{t \in [m, M]} \left[\frac{M - t}{M - m} \bigvee_{m}^{t}(f) + \frac{t - m}{M - m} \bigvee_{t}^{M}(f) \right] \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle\right)$$

$$\leq \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle\right) \bigvee_{m}^{M}(f) \leq \bigvee_{m}^{M}(f) \|x\| \|y\|$$

for any $x, y \in H$.

Proof. First of all, observe that

$$(M - m)\Phi_f(t) = (t - M)[f(t) - f(m)] + (t - m)[f(M) - f(t)] \qquad (5.59)$$

$$= (t - M) \int_{m}^{t} df(s) + (t - m) \int_{t}^{M} df(s)$$

for any $t \in [m, M]$.

Therefore

$$|\Phi_f(t)| \leq \frac{M-t}{M-m}\left|\int_m^t df(s)\right| + \frac{t-m}{M-m}\left|\int_t^M df(s)\right| \qquad (5.60)$$

$$\leq \frac{M-t}{M-m}\bigvee_m^t(f) + \frac{t-m}{M-m}\bigvee_t^M(f)$$

$$\leq \max\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\}\left[\bigvee_m^t(f) + \bigvee_t^M(f)\right]$$

$$= \left[\frac{1}{2} + \frac{\left|t - \frac{m+M}{2}\right|}{M-m}\right]\bigvee_m^M(f)$$

for any $t \in [m, M]$, which implies that

$$\max_{t\in[m,M]}|\Phi_f(t)| \leq \max_{t\in[m,M]}\left[\frac{M-t}{M-m}\bigvee_m^t(f) + \frac{t-m}{M-m}\bigvee_t^M(f)\right] \qquad (5.61)$$

$$\leq \max_{t\in[m,M]}\left[\frac{1}{2} + \frac{\left|t - \frac{m+M}{2}\right|}{M-m}\right]\bigvee_m^M(f) = \bigvee_m^M(f).$$

On applying the inequality (5.56) for the representation (5.55) we have from (5.61) that

$$\left|\int_{m-0}^M \Phi_f(t)\, d(\langle E_t x, y\rangle)\right|$$

$$\leq \max_{t\in[m,M]}\left[\frac{M-t}{M-m}\bigvee_m^t(f) + \frac{t-m}{M-m}\bigvee_t^M(f)\right]\bigvee_m^M(\langle E_{(\cdot)}x, y\rangle)$$

$$\leq \bigvee_m^M(f)\bigvee_m^M(\langle E_{(\cdot)}x, y\rangle)$$

for any $x, y \in H$, which produces the desired result (5.58). ∎

The case of Lipschitzian functions is as follows:

Theorem 347 (Dragomir, 2010, [7]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then we have the inequal-*

ities:

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \quad (5.62)$$

$$\leq \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle)$$

$$\times \max_{t \in [m, M]} \left[\frac{M - t}{M - m} |f(t) - f(m)| + \frac{t - m}{M - m} |f(M) - f(t)| \right]$$

$$\leq \frac{1}{2} (M - m) L \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \leq \frac{1}{2} (M - m) L \|x\| \|y\|$$

for any $x, y \in H$.

Proof. We have from the first part of the equality (5.59) that

$$|\Phi_f(t)| \leq \frac{M - t}{M - m} |f(t) - f(m)| + \frac{t - m}{M - m} |f(M) - f(t)| \quad (5.63)$$

$$\leq \frac{2L}{M - m} (M - t)(t - m) \leq \frac{1}{2} (M - m) L$$

for any $t \in [m, M]$, which, by a similar argument to the one from the above Theorem 346, produces the desired result (5.62). The details are omitted. ∎

The following corollary holds:

Corollary 348 (Dragomir, 2010, [7]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $l, L \in \mathbb{R}$ are such that $L > l$ and $f : [m, M] \to \mathbb{R}$ is $(l, L)-Lipschitzian$ on $[m, M]$, then we have the inequalities:*

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \quad (5.64)$$

$$\leq \frac{1}{4} (M - m)(L - l) \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \leq \frac{1}{4} (M - m)(L - l) \|x\| \|y\|$$

for any $x, y \in H$.

Proof. Follows by applying the inequality (5.62) to the $\frac{1}{2}(L - l)$-Lipschitzian function $f - \frac{1}{2}(l + L) e$, where $e(t) = t$, $t \in [m, M]$. The details are omitted. ∎

When the generating function is continuous convex, we can state the following result as well:

Theorem 349 (Dragomir, 2010, [7]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is continuous convex on $[m, M]$ with finite lateral derivatives $f'_-(M)$ and $f'_+(m)$, then we have the inequalities:*

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \quad (5.65)$$

$$\leq \frac{1}{4}(M - m) \left[f'_-(M) - f'_+(m) \right] \bigvee_m^M \left(\langle E_{(\cdot)}x, y \rangle \right)$$

$$\leq \frac{1}{4}(M - m) \left[f'_-(M) - f'_+(m) \right] \|x\| \|y\|$$

for any $x, y \in H$.

Proof. By the convexity of f on $[m, M]$ we have

$$f(t) - f(M) \geq f'_-(M)(t - M)$$

for any $t \in [m, M]$. If we multiply this inequality with $t - m \geq 0$ we deduce

$$(t - m)f(t) - (t - m)f(M) \geq f'_-(M)(t - M)(t - m) \quad (5.66)$$

for any $t \in [m, M]$.

Similarly, we get

$$(M - t)f(t) - (M - t)f(m) \geq f'_+(m)(M - t)(t - m) \quad (5.67)$$

for any $t \in [m, M]$.

Summing the above inequalities and dividing by $M - m$ we deduce the inequality

$$\Phi_f(t) \leq \frac{(M - t)(t - m)}{M - m} \left[f'_-(M) - f'_+(m) \right] \quad (5.68)$$

$$\leq \frac{1}{4}(M - m) \left[f'_-(M) - f'_+(m) \right]$$

for any $t \in [m, M]$.

By the convexity of f we also have that

$$\frac{1}{M - m} \left[(M - t)f(m) + (t - m)f(M) \right] \quad (5.69)$$

$$\geq f \left(\frac{(M - t)m + (t - m)M}{M - m} \right)$$

$$= f(t)$$

giving that

$$\Phi_f(t) \geq 0 \text{ for any } t \in [m, M]. \quad (5.70)$$

Utilising (5.56) for the representation (5.55) we deduce from (5.68) and (5.70) the desired result (5.65). ∎

5.5.2 Inequalities in the Operator Order

The following result providing some inequalities in the operator order may be stated:

Theorem 350 (Dragomir, 2010, [7]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$.*

1. If $f : [m, M] \to \mathbb{R}$ is continuous on $[m, M]$, then

$$\left| \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A) \right| \tag{5.71}$$

$$\leq \left[\max_{t \in [m,M]} f(t) - \min_{t \in [m,M]} f(t) \right] 1_H.$$

2. If $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then

$$\left| \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A) \right| \tag{5.72}$$

$$\leq \frac{M1_H - A}{M - m} \bigvee_m^A (f) + \frac{A - m1_H}{M - m} \bigvee_A^M (f)$$

$$\leq \left[\frac{1}{2} + \frac{\left| A - \frac{m+M}{2} 1_H \right|}{M - m} \right] \bigvee_m^M (f),$$

where $\bigvee_m^A (f)$ denotes the operator generated by the scalar function $[m, M] \ni t \longmapsto \bigvee_m^t (f) \in \mathbb{R}$. The same notation applies for $\bigvee_A^M (f)$.

3. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then

$$\left| \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A) \right| \tag{5.73}$$

$$\leq \frac{M1_H - A}{M - m} |f(A) - f(m)1_H| + \frac{A - m1_H}{M - m} |f(M)1_H - f(A)|$$

$$\leq \frac{1}{2}(M - m)L1_H.$$

4. If $f : [m, M] \to \mathbb{R}$ is continuous convex on $[m, M]$ with finite lateral derivatives $f'_-(M)$ and $f'_+(m)$, then we have the inequalities:

$$0 \leq \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A) \qquad (5.74)$$

$$\leq \frac{(M1_H - A)(A - m1_H)}{M - m} \left[f'_-(M) - f'_+(m) \right]$$

$$\leq \frac{1}{4} (M - m) \left[f'_-(M) - f'_+(m) \right] 1_H.$$

Proof. Follows by applying the property (P) to the scalar inequalities (5.57), (5.60), (5.63), (5.68) and (5.70). The details are omitted. ∎

The following particular case is perhaps more useful for applications:

Corollary 351 (Dragomir, 2010, [7]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $l, L \in \mathbb{R}$ with $L > l$ and $f : [m, M] \to \mathbb{R}$ is $(l, L)-$Lipschitzian on $[m, M]$, then we have the inequalities:*

$$\left| \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A) \right| \qquad (5.75)$$

$$\leq \frac{1}{4} (M - m)(L - l) 1_H.$$

5.5.3   More Inequalities for Differentiable Functions

The following result holds:

Theorem 352 (Dragomir, 2010, [7]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. Assume that the function $f : I \to \mathbb{C}$ with $[m, M] \subset \mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$.*

1. If the derivative f' is continuous and of bounded variation on $[m, M]$, then we have the inequality

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \qquad (5.76)$$

$$\leq \frac{1}{4} (M - m) \bigvee_m^M (f') \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{4} (M - m) \bigvee_m^M (f') \|x\| \|y\|$$

for any $x, y \in H$.

2. If the derivative f' is Lipschitzian with the constant $K > 0$ on $[m, M]$,
then we have the inequality

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \quad (5.77)$$

$$\leq \frac{1}{8} (M - m)^2 K \bigvee_m^M (\langle E_{(\cdot)} x, y \rangle) \leq \frac{1}{8} (M - m)^2 K \|x\| \|y\|$$

for any $x, y \in H$.

Proof. First of all we notice that if $f : [m, M] \to \mathbb{C}$ is absolutely continuous
on $[m, M]$ and such that the derivative f' is Riemann integrable on $[m, M]$,
then we have the following representation in terms of the Riemann-Stieltjes
integral:

$$\Phi_f(t) = \frac{1}{M - m} \int_m^M K(t, s) \, df'(s), \quad t \in [m, M], \quad (5.78)$$

where the kernel $K : [m, M]^2 \to \mathbb{R}$ is given by

$$K(t, s) := \begin{cases} (M - t)(s - m) & \text{if } m \leq s \leq t \\ (t - m)(M - s) & \text{if } t < s \leq M. \end{cases} \quad (5.79)$$

Indeed, since f' is Riemann integrable on $[m, M]$, it follows that the Riemann-
Stieltjes integrals $\int_m^t (s - m) \, df'(s)$ and $\int_t^M (M - s) \, df'(s)$ exist for each
$t \in [m, M]$. Now, integrating by parts in the Riemann-Stieltjes integral, we
have:

$$\int_m^M K(t, s) \, df'(s) = (M - t) \int_m^t (s - m) \, df'(s) + (t - m) \int_t^M (M - s) \, df'(s)$$

$$= (M - t) \left[(s - m) f'(s) \Big|_m^t - \int_m^t f'(s) \, ds \right]$$

$$+ (t - m) \left[(M - s) f'(s) \Big|_t^M - \int_t^M f'(s) \, ds \right]$$

$$= (M - t) [(t - m) f'(t) - (f(t) - f(m))]$$

$$+ (t - m) [- (M - t) f'(t) + f(M) - f(t)]$$

$$= (t - m) [f(M) - f(t)] - (M - t) [f(t) - f(m)]$$

$$= (M - m) \Phi_f(t)$$

for any $t \in [m, M]$, which provides the desired representation (5.78).

Now, utilizing the representation (5.78) and the property (5.56), we have

$$|\Phi_f(t)| \tag{5.80}$$

$$= \frac{1}{M-m} \left| (M-t) \int_m^t (s-m)\, df'(s) + (t-m) \int_t^M (M-s)\, df'(s) \right|$$

$$\leq \frac{1}{M-m} \left[(M-t) \left| \int_m^t (s-m)\, df'(s) \right| + (t-m) \left| \int_t^M (M-s)\, df'(s) \right| \right]$$

$$\leq \frac{1}{M-m}$$

$$\times \left[(M-t) \bigvee_m^t (f') \sup_{s\in[m,t]} (s-m) + (t-m) \bigvee_t^M (f') \sup_{s\in[t,M]} (M-s) \right]$$

$$= \frac{(t-m)(M-t)}{M-m} \left[\bigvee_m^t (f') + \bigvee_t^M (f') \right]$$

$$= \frac{(t-m)(M-t)}{M-m} \bigvee_m^M (f') \leq \frac{1}{4}(M-m) \bigvee_m^M (f')$$

for any $t \in [m, M]$.

On making use of the representation (5.55) we deduce the desired result (5.76).

Further, we utilize the fact that for an $L-$Lipschitzian function, $p : [\alpha, \beta] \to \mathbb{C}$ and a Riemann integrable function $v : [\alpha, \beta] \to \mathbb{C}$, the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\, dv(s)$ exists and

$$\left| \int_\alpha^\beta p(s)\, dv(s) \right| \leq L \int_\alpha^\beta |p(s)|\, ds.$$

Then, by utilizing (5.80) we have

$$|\Phi_f(t)| \tag{5.81}$$

$$\leq \frac{1}{M-m} \left[(M-t) \left| \int_m^t (s-m)\, df'(s) \right| + (t-m) \left| \int_t^M (M-s)\, df'(s) \right| \right]$$

$$\leq \frac{K}{M-m} \left[(M-t) \int_m^t (s-m)\, ds + (t-m) \int_t^M (M-s)\, ds \right]$$

$$= \frac{K}{M-m} \left[\frac{(M-t)(t-m)^2}{2} + \frac{(t-m)(M-t)^2}{2} \right]$$

$$= \frac{1}{2}(M-m)(t-m)(M-t) K \leq \frac{1}{8}(M-m)^2 K$$

for any $t \in [m, M]$.

On making use of the representation (5.55) we deduce the desired result (5.77). ∎

The following inequalities in the operator order are of interest as well:

Theorem 353 (Dragomir, 2010, [7]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. Assume that the function $f : I \to \mathbb{C}$ with $[m, M] \subset \mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$.*

1. If the derivative f' is continuous and of bounded variation on $[m, M]$, then we have the inequality

$$\left| \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A) \right| \tag{5.82}$$

$$\leq \frac{(A - m1_H)(M1_H - A)}{M - m} \bigvee_m^M (f') \leq \frac{1}{4} (M - m) \bigvee_m^M (f') 1_H.$$

2. If the derivative f' is Lipschitzian with the constant $K > 0$ on $[m, M]$, then we have the inequality

$$\left| \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A) \right| \tag{5.83}$$

$$\leq \frac{1}{2} (M - m)(A - m1_H)(M1_H - A) K \leq \frac{1}{8} (M - m)^2 K 1_H.$$

5.5.4 Applications for Particular Functions

It is obvious that the above results can be applied for various particular functions. However, we will restrict here only to the power and logarithmic functions.

1. Consider now the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^p$ with $p \neq 0$. On applying Theorem 349 we can state the following proposition:

Proposition 354 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$. Then for any $x, y \in H$ we have the inequalities*

$$\left| \left\langle \left[\frac{m^p (M1_H - A) + M^p (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle A^p x, y \rangle \right| \tag{5.84}$$

$$\leq \frac{1}{2} (M - m) \Delta_p \|x\| \|y\|$$

where

$$\Delta_p = p \times \begin{cases} M^{p-1} - m^{p-1} & \text{if } p \in (-\infty, 0) \cup [1, \infty) \\ m^{p-1} - M^{p-1} & \text{if } 0 < p < 1. \end{cases}$$

In particular,

$$\left| \left\langle \left[\frac{M\left(M1_H - A\right) + m\left(A - m1_H\right)}{mM\left(M - m\right)} \right] x, y \right\rangle - \left\langle A^{-1}x, y \right\rangle \right| \qquad (5.85)$$

$$\leq \frac{1}{2} \frac{\left(M - m\right)^2 \left(M + m\right)}{m^2 M^2} \|x\| \, \|y\|$$

for any $x, y \in H$.

The following inequalities in the operator order also hold:

Proposition 355 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\left(A\right) \subseteq [m, M]$ for some real numbers $0 < m < M$.*
  If $p \in (-\infty, 0) \cup [1, \infty)$, then

$$0 \leq \frac{m^p \left(M1_H - A\right) + M^p \left(A - m1_H\right)}{M - m} - A^p \qquad (5.86)$$

$$\leq p \frac{\left(M1_H - A\right) \left(A - m1_H\right)}{M - m} \left(M^{p-1} - m^{p-1}\right)$$

$$\leq \frac{1}{4} p \left(M - m\right) \left(M^{p-1} - m^{p-1}\right) 1_H.$$

If $p \in (0, 1)$, then

$$0 \leq A^p - \frac{m^p \left(M1_H - A\right) + M^p \left(A - m1_H\right)}{M - m} \qquad (5.87)$$

$$\leq p \frac{\left(M1_H - A\right) \left(A - m1_H\right)}{M - m} \left(m^{p-1} - M^{p-1}\right)$$

$$\leq \frac{1}{4} p \left(M - m\right) \left(m^{p-1} - M^{p-1}\right) 1_H.$$

In particular, we have the inequalities

$$0 \leq \frac{M\left(M1_H - A\right) + m\left(A - m1_H\right)}{mM\left(M - m\right)} - A^{-1} \qquad (5.88)$$

$$\leq \frac{\left(M1_H - A\right) \left(A - m1_H\right)}{M - m} \cdot \frac{M^2 - m^2}{m^2 M^2}$$

$$\leq \frac{1}{2} \frac{\left(M - m\right)^2 \left(M + m\right)}{m^2 M^2} 1_H.$$

The proof follows from (5.74) and the details are omitted.
2. The case of logarithmic function is as follows:

Proposition 356 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\left(A\right) \subseteq [m, M]$ for some real numbers $0 < m < M$.*

Then for any $x, y \in H$ we have the inequalities

$$\left| \left\langle \left[\frac{(M1_H - A) \ln m + (A - m1_H) \ln M}{M - m} \right] x, y \right\rangle - \langle \ln Ax, y \rangle \right| \qquad (5.89)$$

$$\leq \frac{1}{4} \frac{(M - m)^2}{mM} \| x \| \, \| y \| \, .$$

We also have the following inequality in the operator order

$$0 \leq \ln A - \frac{(M1_H - A) \ln m + (A - m1_H) \ln M}{M - m} \qquad (5.90)$$

$$\leq \frac{(M1_H - A)(A - m1_H)}{Mm} \leq \frac{1}{4} \frac{(M - m)^2}{mM} 1_H.$$

Remark 357 *Similar results can be obtained if ones uses the inequalities from Theorem 352 and 353. However the details are left to the interested reader.*

5.6   Product Inequalities

5.6.1   Some Vector Inequalities

In this section we investigate the quantity

$$| \langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, y \rangle |$$

where x, y are vectors in the Hilbert space H and A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$, and provide different bounds for some classes of continuous functions $f : [m, M] \to \mathbb{C}$. Applications for some particular cases including the power and logarithmic functions are provided as well.

The following representation in terms of the spectral family is of interest in itself:

Lemma 358 (Dragomir, 2010, [8]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$ with $f(M) \neq f(m)$, then we have the representation*

$$\frac{1}{[f(M) - f(m)]^2} [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] \qquad (5.91)$$

$$= \frac{1}{f(M) - f(m)}$$

$$\times \int_{m-0}^{M} \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s) \right) \left(E_t - \frac{1}{2} 1_H \right) df(t) \, .$$

Proof. We observe that,

$$\frac{1}{f(M) - f(m)} \int_{m-0}^{M} \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s) \right) \quad (5.92)$$

$$\times \left(E_t - \frac{1}{2} 1_H \right) df(t)$$

$$= \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t^2 df(t)$$

$$- \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s) \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t df(t)$$

$$- \frac{1}{2} \int_{m-0}^{M} E_t df(t) + \frac{1}{2} \int_{m-0}^{M} E_s df(s)$$

$$= \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t^2 df(t) - \left[\frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t df(t) \right]^2$$

which is an equality of interest in itself.

Since E_t are projections, we have $E_t^2 = E_t$ for any $t \in [m, M]$ and then we can write that

$$\frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t^2 df(t) - \left[\frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t df(t) \right]^2 \quad (5.93)$$

$$= \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t df(t) - \left[\frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t df(t) \right]^2$$

$$= \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t df(t) \left[1_H - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t df(t) \right].$$

Integrating by parts in the Riemann-Stieltjes integral and utilizing the spectral representation theorem we have

$$\int_{m-0}^{M} E_t df(t) = f(M) 1_H - f(A)$$

and

$$1_H - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_t df(t) = \frac{f(A) - f(m) 1_H}{f(M) - f(m)},$$

which together with (5.93) and (5.92) produce the desired result (5.91). ∎

The following vector version may be stated as well:

Corollary 359 (Dragomir, 2010, [8]) *With the assumptions of Lemma 358 we have the equality*

$$\langle [f(M) 1_H - f(A)][f(A) - f(m) 1_H] x, y \rangle \tag{5.94}$$
$$= [f(M) - f(m)]$$
$$\times \int_{m-0}^{M} \left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s) \right) x, \left(E_t - \frac{1}{2} 1_H \right) y \right\rangle df(t),$$

for any $x, y \in [m, M]$.

The following result that provides some bounds for continuous functions of bounded variation may be stated as well:

Theorem 360 (Dragomir, 2010, [8]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$ with $f(M) \neq f(m)$, then we have the inequality*

$$|\langle [f(M) 1_H - f(A)][f(A) - f(m) 1_H] x, y \rangle| \tag{5.95}$$
$$\leq \frac{1}{2} \|y\| |f(M) - f(m)| \bigvee_{m}^{M}(f)$$
$$\times \sup_{t \in [m, M]} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s) \right\| \leq \frac{1}{2} \|x\| \|y\| \left[\bigvee_{m}^{M}(f) \right]^2,$$

for any $x, y \in H$.

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is a bounded function, $v : [a, b] \to \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t) dv(t)$ exists, then the following inequality holds

$$\left| \int_a^b p(t) dv(t) \right| \leq \sup_{t \in [a,b]} |p(t)| \bigvee_a^b(v), \tag{5.96}$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a, b]$.

Utilising this property and the representation (5.94) we have by the Schwarz inequality in Hilbert space H that

$$|\langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, y \rangle| \tag{5.97}$$

$$\leq |f(M) - f(m)| \bigvee_{m}^{M} (f)$$

$$\times \sup_{t \in [m,M]} \left| \left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s) \right) x, \left(E_t - \frac{1}{2} 1_H \right) y \right\rangle \right|$$

$$\leq |f(M) - f(m)| \bigvee_{m}^{M} (f)$$

$$\times \sup_{t \in [m,M]} \left[\left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\| \left\| E_t y - \frac{1}{2} y \right\| \right]$$

for any $x, y \in [m, M]$.

Since E_t are projections, and in this case we have

$$\left\| E_t y - \frac{1}{2} y \right\|^2 = \|E_t y\|^2 - \langle E_t y, y \rangle + \frac{1}{4} \|y\|^2$$

$$= \langle E_t^2 y, y \rangle - \langle E_t y, y \rangle + \frac{1}{4} \|y\|^2 = \frac{1}{4} \|y\|^2,$$

then from (5.97) we deduce the first part of (5.95).

Now, by the same property (5.96) for vector valued functions p with values in Hilbert spaces, we also have that

$$\left\| [f(M) - f(m)] E_t x - \int_{m-0}^{M} E_s x df(s) \right\| \tag{5.98}$$

$$= \left\| \int_{m-0}^{M} (E_t x - E_s x) df(s) \right\| \leq \bigvee_{m}^{M} (f) \sup_{s \in [m,M]} \|E_t x - E_s x\|$$

for any $t \in [m, M]$ and $x \in H$.

Since $0 \leq E_t \leq 1_H$ in the operator order, then $-1_H \leq E_t - E_s \leq 1$ which gives that $-\|x\|^2 \leq \langle (E_t - E_s) x, x \rangle \leq \|x\|^2$, i.e., $|\langle (E_t - E_s) x, x \rangle| \leq \|x\|^2$ for any $x \in H$, which implies that $\|E_t - E_s\| \leq 1$ for any $t, s \in [m, M]$. Therefore $\sup_{s \in [m,M]} \|E_t x - E_s x\| \leq \|x\|$ which together with (5.98) prove the last part of (5.95). ∎

The case of Lipschitzian functions is as follows:

Theorem 361 (Dragomir, 2010, [8]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$*

is a Lipschitzian function with the constant $L > 0$ on $[m, M]$ and with $f(M) \neq f(m)$, then we have the inequality

$$|\langle [f(M) 1_H - f(A)][f(A) - f(m) 1_H] x, y \rangle| \tag{5.99}$$
$$\leq \frac{1}{2} L \|y\| |f(M) - f(m)|$$
$$\times \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x \, df(s) \right\| dt$$
$$\leq \frac{1}{2} L^2 \|y\| \int_{m-0}^{M} \int_{m-0}^{M} \|E_t x - E_s x\| \, ds \, dt$$
$$\leq \frac{\sqrt{2}}{2} L^2 \|y\| (M - m) \langle Ax - mx, Mx - Ax \rangle^{1/2} \leq \frac{\sqrt{2}}{4} L^2 \|y\| \|x\| (M - m)^2$$

for any $x, y \in H$.

Proof. Recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq L \int_a^b |p(t)| \, dt. \tag{5.100}$$

Now, on applying this property of the Riemann-Stieltjes integral, then we have from the representation (5.94) that

$$|\langle [f(M) 1_H - f(A)][f(A) - f(m) 1_H] x, y \rangle| \tag{5.101}$$
$$\leq |f(M) - f(m)|$$
$$\times \int_{m-0}^{M} \left| \left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s \, df(s) \right) x, \left(E_t - \frac{1}{2} 1_H \right) y \right\rangle \right| df(t),$$
$$\leq L |f(M) - f(m)|$$
$$\times \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x \, df(s) \right\| \left\| E_t y - \frac{1}{2} y \right\| dt$$
$$= \frac{1}{2} L \|y\| |f(M) - f(m)|$$
$$\times \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x \, df(s) \right\| dt$$

for any $x, y \in H$ and the first inequality in (5.99) is proved.

Further, observe that

$$|f(M) - f(m)| \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x \, df(s) \right\| dt \quad (5.102)$$

$$= \int_{m-0}^{M} \left\| [f(M) - f(m)] E_t x - \int_{m-0}^{M} E_s x \, df(s) \right\| dt$$

$$= \int_{m-0}^{M} \left\| \int_{m-0}^{M} (E_t x - E_s x) \, df(s) \right\| dt$$

for any $x \in H$.

If we use the vector valued version of the property (5.100), then we have

$$\int_{m-0}^{M} \left\| \int_{m-0}^{M} (E_t x - E_s x) \, df(s) \right\| dt \leq L \int_{m-0}^{M} \int_{m-0}^{M} \| E_t x - E_s x \| \, ds \, dt$$

$$(5.103)$$

for any $x \in H$ and the second part of (5.99) is proved.

Further on, by applying the double integral version of the Cauchy-Buniakowski-Schwarz inequality we have

$$\int_{m-0}^{M} \int_{m-0}^{M} \| E_t x - E_s x \| \, ds \, dt \quad (5.104)$$

$$\leq (M - m) \left(\int_{m-0}^{M} \int_{m-0}^{M} \| E_t x - E_s x \|^2 \, ds \, dt \right)^{1/2}$$

for any $x \in H$.

Now, by utilizing the fact that E_s are projections for each $s \in [m, M]$, then we have

$$\int_{m-0}^{M} \int_{m-0}^{M} \| E_t x - E_s x \|^2 \, ds \, dt \quad (5.105)$$

$$= 2 \left[(M - m) \int_{m-0}^{M} \| E_t x \|^2 \, dt - \left\| \int_{m-0}^{M} E_t x \, dt \right\|^2 \right]$$

$$= 2 \left[(M - m) \int_{m-0}^{M} \langle E_t x, x \rangle \, dt - \left\| \int_{m-0}^{M} E_t x \, dt \right\|^2 \right]$$

for any $x \in H$.

If we integrate by parts and use the spectral representation theorem, then we get

$$\int_{m-0}^{M} \langle E_t x, x \rangle \, dt = \langle Mx - Ax, x \rangle \quad \text{and} \quad \int_{m-0}^{M} E_t x \, dt = Mx - Ax$$

and by (5.105) we then obtain the following equality of interest

$$\int_{m-0}^{M} \int_{m-0}^{M} \|E_t x - E_s x\|^2 \, ds dt = 2 \langle Ax - mx, Mx - Ax \rangle \qquad (5.106)$$

for any $x \in H$.

On making use of (5.106) and (5.104) we then deduce the third part of (5.99).

Finally, by utilizing the elementary inequality in inner product spaces

$$\operatorname{Re} \langle a, b \rangle \leq \frac{1}{4} \|a + b\|^2, \ a, b \in H, \qquad (5.107)$$

we also have that

$$\langle Ax - mx, Mx - Ax \rangle \leq \frac{1}{4} (M - m)^2 \|x\|^2$$

for any $x \in H$, which proves the last inequality in (5.99). ∎

The case of nondecreasing monotonic functions is as follows:

Theorem 362 (Dragomir, 2010, [8]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a monotonic nondecreasing function on $[m, M]$, then we have the inequality*

$$|\langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, y \rangle| \qquad (5.108)$$

$$\leq \frac{1}{2} \|y\| [f(M) - f(m)]$$

$$\times \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\| df(t)$$

$$\leq \frac{1}{2} \|y\| [f(M) - f(m)]$$

$$\times \langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, x \rangle^{1/2}$$

$$\leq \frac{1}{4} \|y\| \|x\| [f(M) - f(m)]^2$$

for any $x, y \in H$.

Proof. From the theory of Riemann-Stieltjes integral it is also well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t) \, dv(t)$ and $\int_a^b |p(t)| \, dv(t)$ exist and

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \int_a^b |p(t)| \, dv(t).$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (5.94) that

$$|\langle [f(M) 1_H - f(A)][f(A) - f(m) 1_H] x, y \rangle| \qquad (5.109)$$
$$\leq [f(M) - f(m)]$$
$$\times \int_{m-0}^{M} \left| \left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s) \right) x, \left(E_t - \frac{1}{2} 1_H \right) y \right\rangle \right| df(t),$$
$$\leq [f(M) - f(m)]$$
$$\times \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\| \left\| E_t y - \frac{1}{2} y \right\| df(t)$$
$$= \frac{1}{2} \|y\| [f(M) - f(m)]$$
$$\times \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\| df(t)$$

for any $x, y \in H$, which proves the first inequality in (5.108).

On utilizing the Cauchy-Buniakowski-Schwarz type inequality for the Riemann-Stieltjes integral of monotonic nondecreasing integrators, we have

$$\int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\| df(t) \qquad (5.110)$$
$$\leq \left[\int_{m-0}^{M} df(t) \right]^{1/2}$$
$$\times \left[\int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2 df(t) \right]^{1/2}$$

for any $x, y \in H$.

Observe that

$$\int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2 df(t) \tag{5.111}$$

$$= \int_{m-0}^{M} \left[\|E_t x\|^2 - 2 \operatorname{Re} \left\langle E_t x, \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\rangle \right.$$

$$+ \left. \left\| \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2 \right] df(t)$$

$$= [f(M) - f(m)] \left[\frac{1}{f(M) - f(m)} \int_{m-0}^{M} \|E_t x\|^2 df(t) \right.$$

$$- \left. \left\| \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2 \right]$$

and, integrating by parts in the Riemann-Stieltjes integral, we have

$$\int_{m-0}^{M} \|E_t x\|^2 df(t) = \int_{m-0}^{M} \langle E_t x, E_t x \rangle df(t) \tag{5.112}$$

$$= \int_{m-0}^{M} \langle E_t x, x \rangle df(t)$$

$$= f(M) \|x\|^2 - \int_{m-0}^{M} f(t) d \langle E_t x, x \rangle$$

$$= f(M) \|x\|^2 - \langle f(A) x, x \rangle = \langle [f(M) 1_H - f(A)] x, x \rangle$$

and

$$\int_{m-0}^{M} E_s x df(s) = f(M) x - f(A) x \tag{5.113}$$

for any $x \in H$.

On making use of the equalities (5.112) and (5.113) we have

$$\frac{1}{f(M) - f(m)} \int_{m-0}^{M} \|E_t x\|^2 df(t) \tag{5.114}$$

$$- \left\| \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2$$

$$= \frac{1}{[f(M) - f(m)]^2}$$

$$\times \left[[f(M) - f(m)] \langle [f(M) 1_H - f(A)] x, x \rangle - \|f(M) x - f(A) x\|^2 \right]$$

$$= \frac{\langle f(M) x - f(A) x, f(A) x - f(m) x \rangle}{[f(M) - f(m)]^2}$$

for any $x \in H$.

Therefore, we obtain the following equality of interest in itself as well

$$\frac{1}{f(M) - f(m)} \tag{5.115}$$

$$\times \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2 df(t)$$

$$= \frac{\langle f(M) x - f(A) x, f(A) x - f(m) x \rangle}{[f(M) - f(m)]^2}$$

$$= \frac{\langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, x \rangle}{[f(M) - f(m)]^2}$$

for any $x \in H$

On making use of the inequality (5.110) we deduce the second inequality in (5.108).

The last part follows by (5.107) and the details are omitted. ∎

5.6.2 Applications

We consider the power function $f(t) := t^p$ where $p \in \mathbb{R} \setminus \{0\}$ and $t > 0$. The following power inequalities hold:

Proposition 363 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 \le m < M$.*
If $p > 0$, then for any $x, y \in H$

$$|\langle (M^p 1_H - A^p)(A^p - m^p 1_H) x, y \rangle| \tag{5.116}$$

$$\le \frac{\sqrt{2}}{2} B_p^2 \|y\| (M - m) \langle Ax - mx, Mx - Ax \rangle^{1/2}$$

$$\le \frac{\sqrt{2}}{4} B_p^2 \|y\| \|x\| (M - m)^2$$

where

$$B_p = p \times \begin{cases} M^{p-1} & \text{if } p \ge 1 \\ m^{p-1} & \text{if } 0 < p < 1, m > 0 \end{cases}$$

and

$$|\langle (A^{-p} - M^{-p} 1_H)(m^{-p} 1_H - A^{-p}) x, y \rangle| \tag{5.117}$$

$$\le \frac{\sqrt{2}}{2} C_p^2 \|y\| (M - m) \langle Ax - mx, Mx - Ax \rangle^{1/2}$$

$$\le \frac{\sqrt{2}}{4} C_p^2 \|y\| \|x\| (M - m)^2,$$

where

$$C_p = pm^{-p-1} \text{ and } m > 0.$$

The proof follows from Theorem 361 applied for the power function.

Proposition 364 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\,(A) \subseteq [m, M]$ for some real numbers with $0 \le m < M$. If $p > 0$, then for any $x, y \in H$*

$$|\langle (M^p 1_H - A^p)(A^p - m^p 1_H) x, y \rangle| \qquad (5.118)$$

$$\le \frac{1}{2} \|y\| (M^p - m^p) \langle (M^p 1_H - A^p)(A^p - m^p 1_H) x, x \rangle^{1/2}$$

$$\le \frac{1}{4} \|y\| \|x\| (M^p - m^p)^2$$

and

$$|\langle (A^{-p} - M^{-p} 1_H)(m^{-p} 1_H - A^{-p}) x, y \rangle| \qquad (5.119)$$

$$\le \frac{1}{2} \|y\| (m^{-p} - M^{-p}) \langle (A^{-p} - M^{-p} 1_H)(m^{-p} 1_H - A^{-p}) x, x \rangle^{1/2}$$

$$\le \frac{1}{4} \|y\| \|x\| (m^{-p} - M^{-p})^2 .$$

The proof follows from Theorem 362.

Now, consider the logarithmic function $f(t) = \ln t, t > 0$. We have

Proposition 365 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\,(A) \subseteq [m, M]$ for some real numbers with $0 < m < M$. Then we have the inequalities*

$$|\langle [(\ln M) 1_H - \ln A][\ln A - (\ln m) 1_H] x, y \rangle| \qquad (5.120)$$

$$\le \frac{\sqrt{2}}{2m^2} \|y\| (M - m) \langle Ax - mx, Mx - Ax \rangle^{1/2}$$

$$\le \frac{\sqrt{2}}{4} \|y\| \|x\| \left(\frac{M}{m} - 1 \right)^2$$

and

$$|\langle [(\ln M) 1_H - \ln A][\ln A - (\ln m) 1_H] x, y \rangle| \qquad (5.121)$$

$$\le \frac{1}{2} \|y\| \langle [(\ln M) 1_H - \ln A][\ln A - (\ln m) 1_H] x, x \rangle^{1/2} \ln \left(\frac{M}{m} \right)$$

$$\le \frac{1}{4} \|y\| \|x\| \left[\ln \left(\frac{M}{m} \right) \right]^2 .$$

The proof follows from Theorem 361 and 362 applied for the logarithmic function.

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6
Inequalities of Taylor Type

6.1 Introduction

In approximating n-time differentiable functions around a point, perhaps
the classical Taylor's expansion is one of the simplest and most convenient
and elegant methods that has been employed in the development of Math-
ematics for the last three centuries. There is probably no field of Science
where Mathematical Modelling is used not to contain in a form or another
Taylor's expansion for functions that are differentiable in a certain sense.

In the present chapter, that is intended to be developed to a later stage,
we present some error bounds in approximating n-time differentiable func-
tions of selfadjoint operators by the use of operator Taylor's type expansions
around a point or two points from its spectrum for which the remainder is
known in an integral form.

Some applications for elementary functions including the exponential and
logarithmic functions are provided as well.

6.2 Taylor's Type Inequalities

6.2.1 Some Identities

In this section, by utilizing the spectral representation theorem of self-
adjoint operators in Hilbert spaces, some error bounds in approximating
n-time differentiable functions of selfadjoint operators in Hilbert Spaces

via a Taylor's type expansion are given. Applications for some elementary functions of interest including the exponential and logarithmic functions are also provided.

The following result provides a Taylor's type representation for a function of selfadjoint operators in Hilbert spaces with integral remainder.

Theorem 366 (Dragomir, 2010, [5]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the equalities*

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(A - c1_H)^k + R_n(f, c, m, M) \tag{6.1}$$

where

$$R_n(f, c, m, M) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right) dE_\lambda. \tag{6.2}$$

Proof. We utilize the Taylor formula for a function $f : I \to \mathbb{C}$ whose n-th derivative $f^{(n)}$ is locally of bounded variation on the interval I to write the equality

$$f(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(\lambda - c)^k + \frac{1}{n!} \int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \tag{6.3}$$

for any $\lambda, c \in [m, M]$, where the integral is taken in the Riemann-Stieltjes sense.

If we integrate the equality on $[m, M]$ in the Riemann-Stieltjes sense with the integrator E_λ we get

$$\int_{m-0}^{M} f(\lambda) \, dE_\lambda = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \int_{m-0}^{M} (\lambda - c)^k \, dE_\lambda$$

$$+ \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right) dE_\lambda$$

which, by the spectral representation theorem, produces the equality (6.1) with the representation of the remainder from (6.2). ■

The following particular instances are of interest for applications:

Corollary 367 (Dragomir, 2010, [5]) *With the assumptions of the above Theorem 366, we have the equalities*

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(m)(A - m1_H)^k + L_n(f, c, m, M) \tag{6.4}$$

where

$$L_n\left(f, c, m, M\right) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_{m}^{\lambda} \left(\lambda - t\right)^n d\left(f^{(n)}\left(t\right)\right) \right) dE_\lambda$$

and

$$f\left(A\right) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)} \left(\frac{m+M}{2}\right) \left(A - \frac{m+M}{2} 1_H\right)^k \qquad (6.5)$$
$$+ M_n\left(f, c, m, M\right)$$

where

$$M_n\left(f, c, m, M\right) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_{\frac{m+M}{2}}^{\lambda} \left(\lambda - t\right)^n d\left(f^{(n)}\left(t\right)\right) \right) dE_\lambda$$

and

$$f\left(A\right) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}\left(M\right) \left(M 1_H - A\right)^k + U_n\left(f, c, m, M\right) \qquad (6.6)$$

where

$$U_n\left(f, c, m, M\right) = \frac{(-1)^{n+1}}{n!} \int_{m-0}^{M} \left(\int_{\lambda}^{M} \left(t - \lambda\right)^n d\left(f^{(n)}\left(t\right)\right) \right) dE_\lambda, \quad (6.7)$$

respectively.

Remark 368 *We remark that, if the n-th derivative of the function f considered above is absolutely continuous on the interval $[m, M]$, then we have the representation (6.1) with the remainder*

$$R_n\left(f, c, m, M\right) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_{c}^{\lambda} \left(\lambda - t\right)^n f^{(n+1)}\left(t\right) dt \right) dE_\lambda. \qquad (6.8)$$

Here the integral $\int_{c}^{\lambda} \left(\lambda - t\right)^n f^{(n+1)}\left(t\right) dt$ is considered in the Lebesgue sense. Similar representations hold true when c is taken the particular values m, M or $\frac{m+M}{2}$.

Now, if we consider the exponential function, then for any selfadjoint operator A in the Hilbert space H with the spectrum $Sp\left(A\right) \subseteq [m, M]$ and with the spectral family $\{E_\lambda\}_\lambda$ we have the representation

$$e^{A - c1_H} = \sum_{k=0}^{n} \frac{1}{k!} \left(A - c1_H\right)^k + \frac{1}{n!} \int_{m-0}^{M} \left(\int_{c}^{\lambda} \left(\lambda - t\right)^n e^{t-c} dt \right) dE_\lambda, \quad (6.9)$$

where c is any real number.

Further, if we consider the logarithmic function, then for any positive definite operator A with $Sp(A) \subseteq [m, M] \subset (0, \infty)$ and with the spectral family $\{E_\lambda\}_\lambda$ we have

$$\ln A = (\ln c) 1_H + \sum_{k=1}^{n} \frac{(-1)^{k-1}(A - c1_H)^k}{kc^k} \tag{6.10}$$
$$+ (-1)^n \int_{m-0}^{M} \left(\int_{c}^{\lambda} \frac{(\lambda - t)^n}{t^{n+1}} dt \right) dE_\lambda$$

for any $c > 0$.

6.2.2   Some Error Bounds

We start with the following result that provides an approximation for an n-time differentiable function of selfadjoint operators in Hilbert spaces:

Theorem 369 (Dragomir, 2010, [5]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality*

$$|\langle R_n(f, c, m, M) x, y \rangle| \tag{6.11}$$
$$= \left| \langle f(A) x, y \rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \left\langle (A - c1_H)^k x, y \right\rangle \right|$$
$$\leq \frac{1}{n!} \left[(c - m)^n \bigvee_{m}^{c} \left(f^{(n)} \right) \bigvee_{m}^{c} \left(\langle E_{(\cdot)} x, y \rangle \right) \right.$$
$$\left. + (M - c)^n \bigvee_{c}^{M} \left(f^{(n)} \right) \bigvee_{c}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right) \right]$$
$$\leq \frac{1}{n!} \max \left\{ (M - c)^n \bigvee_{c}^{M} \left(f^{(n)} \right), (c - m)^n \bigvee_{c}^{M} \left(f^{(n)} \right) \right\} \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$
$$\leq \frac{1}{n!} \left(\frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^n \bigvee_{m}^{M} \left(f^{(n)} \right) \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right),$$

for any $x, y \in H$.

Proof. From the identities (6.1) and (6.2) we have

$$\langle R_n (f, c, m, M) x, y \rangle \tag{6.12}$$

$$= \langle f(A) x, y \rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \left\langle (A - c1_H)^k x, y \right\rangle$$

$$= \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right) d\langle E_\lambda x, y \rangle$$

$$= \frac{1}{n!} \int_{m-0}^{c} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right) d\langle E_\lambda x, y \rangle$$

$$+ \frac{1}{n!} \int_c^{M} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right) d\langle E_\lambda x, y \rangle$$

for any $x, y \in H$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \le \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v), \tag{6.13}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Taking the modulus in (6.12) and utilizing the inequality (6.13) we have

$$|\langle R_n (f, c, m, M) x, y \rangle| \tag{6.14}$$

$$\le \frac{1}{n!} \left| \int_{m-0}^{c} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right) d\langle E_\lambda x, y \rangle \right|$$

$$+ \frac{1}{n!} \left| \int_c^{M} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right) d\langle E_\lambda x, y \rangle \right|$$

$$\le \frac{1}{n!} \max_{\lambda \in [m,c]} \left| \int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right| \bigvee_m^c \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$+ \frac{1}{n!} \max_{\lambda \in [c,M]} \left| \int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right| \bigvee_c^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

for any $x, y \in H$.

By the same property (6.13) we have

$$\max_{\lambda \in [m,c]} \left| \int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right| \le (c - m)^n \bigvee_m^c \left(f^{(n)} \right) \tag{6.15}$$

and

$$\max_{\lambda \in [c,M]} \left| \int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right| \leq (M - c)^n \bigvee_c^M \left(f^{(n)} \right). \qquad (6.16)$$

Now, on making use of (6.14)-(6.16) we deduce

$$|\langle R_n (f, c, m, M) x, y \rangle|$$

$$\leq \frac{1}{n!} \left[(c - m)^n \bigvee_m^c \left(f^{(n)} \right) \bigvee_m^c \left(\langle E_{(\cdot)} x, y \rangle \right) \right.$$

$$\left. + (M - c)^n \bigvee_c^M \left(f^{(n)} \right) \bigvee_c^M \left(\langle E_{(\cdot)} x, y \rangle \right) \right]$$

$$\leq \frac{1}{n!} \max \left\{ (c - m)^n \bigvee_m^c \left(f^{(n)} \right), (M - c)^n \bigvee_c^M \left(f^{(n)} \right) \right\}$$

$$\times \left[\bigvee_m^c \left(\langle E_{(\cdot)} x, y \rangle \right) + \bigvee_c^M \left(\langle E_{(\cdot)} x, y \rangle \right) \right]$$

$$\leq \frac{1}{n!} \max \left\{ (c - m)^n, (M - c)^n \right\} \bigvee_m^M \left(f^{(n)} \right) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$= \frac{1}{n!} \left(\frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^n \bigvee_m^M \left(f^{(n)} \right) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

for any $x, y \in H$ and the proof is complete. ∎

The following particular cases are of interest for applications

Corollary 370 (Dragomir, 2010, [5]) *With the assumption of Theorem 369 we have the inequalities*

$$\left| \langle f(A) x, y \rangle - \sum_{k=0}^n \frac{1}{k!} f^{(k)}(m) \left\langle (A - m 1_H)^k x, y \right\rangle \right| \qquad (6.17)$$

$$\leq \frac{1}{n!} (M - m)^n \bigvee_m^M \left(f^{(n)} \right) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{n!} (M - m)^n \bigvee_m^M \left(f^{(n)} \right) \|x\| \|y\|,$$

$$\left| \langle f(A) x, y \rangle - \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(M) \left\langle (M1_H - A)^k x, y \right\rangle \right| \qquad (6.18)$$

$$\leq \frac{1}{n!} (M - m)^n \bigvee_{m}^{M} \left(f^{(n)} \right) \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{n!} (M - m)^n \bigvee_{m}^{M} \left(f^{(n)} \right) \|x\| \|y\|$$

and

$$\left| \langle f(A) x, y \rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)} \left(\frac{m+M}{2} \right) \left\langle \left(A - \frac{m+M}{2} 1_H \right)^k x, y \right\rangle \right| \quad (6.19)$$

$$\leq \frac{1}{2^n n!} (M - m)^n \max \left\{ \bigvee_{\frac{m+M}{2}}^{M} \left(f^{(n)} \right), \bigvee_{m}^{\frac{m+M}{2}} \left(f^{(n)} \right) \right\} \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{2^n n!} (M - m)^n \max \left\{ \bigvee_{\frac{m+M}{2}}^{M} \left(f^{(n)} \right), \bigvee_{m}^{\frac{m+M}{2}} \left(f^{(n)} \right) \right\} \|x\| \|y\|$$

respectively, for any $x, y \in H$.

Proof. The first part in the inequalities follow from (6.11) by choosing $c = m, c = M$ and $c = \frac{m+M}{2}$ respectively.

The last part follows by the Total Variation Schwarz's inequality and we omit the details. ∎

The following result also holds:

Theorem 371 (Dragomir, 2010, [5]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality*

$$|\langle R_n(f, c, m, M) x, y \rangle| \qquad (6.20)$$

$$\leq \frac{1}{(n+1)!} L_n \left[(c - m)^{n+1} \bigvee_{m}^{c} \left(\langle E_{(\cdot)} x, y \rangle \right) + (M - c)^{n+1} \bigvee_{c}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right) \right]$$

$$\leq \frac{1}{(n+1)!} L_n \left(\frac{1}{2} (M - m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{(n+1)!} L_n \left(\frac{1}{2} (M - m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \|x\| \|y\|$$

for any $x, y \in H$.

Proof. First of all, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\, dv(t) \right| \leq L \int_a^b |p(t)|\, dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have

$$\max_{\lambda \in [m,c]} \left| \int_\lambda^c (t - \lambda)^n\, d\left(f^{(n)}(t) \right) \right| \leq \max_{\lambda \in [m,c]} \left[L_n \int_\lambda^c (t - \lambda)^n\, dt \right] \quad (6.21)$$

$$= \frac{L_n}{n+1} (c - m)^{n+1}$$

and

$$\max_{\lambda \in [c,M]} \left| \int_c^\lambda (\lambda - t)^n\, d\left(f^{(n)}(t) \right) \right| \leq \max_{\lambda \in [c,M]} \left[L_n \int_c^\lambda (\lambda - t)^n\, dt \right] \quad (6.22)$$

$$= \frac{L_n}{n+1} (M - c)^{n+1}.$$

Now, on utilizing the inequality (6.14), then we have from (6.21) and (6.22) that

$$|\langle R_n(f, c, m, M) x, y \rangle| \quad (6.23)$$

$$\leq \frac{1}{(n+1)!} L_n (c - m)^{n+1} \bigvee_m^c \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$+ \frac{1}{(n+1)!} L_n (M - c)^{n+1} \bigvee_c^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{(n+1)!} L_n \max \left\{ (c - m)^{n+1}, (M - c)^{n+1} \right\} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$= \frac{1}{(n+1)!} L_n \left(\frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right)^{n+1} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right),$$

and the proof is complete. ∎

The following particular cases are of interest for applications:

Corollary 372 (Dragomir, 2010, [5]) *With the assumption of Theorem 371 we have the inequalities*

$$\left| \langle f(A) x, y \rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(m) \left\langle (A - m 1_H)^k x, y \right\rangle \right| \qquad (6.24)$$

$$\leq \frac{1}{(n+1)!} (M-m)^{n+1} L_n \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{(n+1)!} (M-m)^{n+1} L_n \|x\| \|y\|$$

and

$$\left| \langle f(A) x, y \rangle - \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(M) \left\langle (M 1_H - A)^k x, y \right\rangle \right| \qquad (6.25)$$

$$\leq \frac{1}{(n+1)!} (M-m)^{n+1} L_n \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{(n+1)!} (M-m)^{n+1} L_n \|x\| \|y\|$$

and

$$\left| \langle f(A) x, y \rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)} \left(\frac{m+M}{2} \right) \left\langle \left(A - \frac{m+M}{2} 1_H \right)^k x, y \right\rangle \right| \quad (6.26)$$

$$\leq \frac{1}{2^{n+1}(n+1)!} (M-m)^{n+1} L_n \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{2^{n+1}(n+1)!} (M-m)^{n+1} L_n \|x\| \|y\|$$

respectively, for any $x, y \in H$.

The following corollary that provides a perturbed version of Taylor's expansion holds:

Corollary 373 (Dragomir, 2010, [5]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $g : I \to \mathbb{R}$ is such that the n-th derivative $g^{(n)}$ is (l_n, L_n)−Lipschitzian with the constant $L_n > l_n > 0$ on the interval $[m, M]$, then for any $c \in [m, M]$*

we have the inequality

$$\left| \langle g\left(A\right) x, y \rangle - g\left(c\right) \langle x, y \rangle - \sum_{k=1}^{n} \frac{1}{k!} g^{(k)}\left(c\right) \left\langle \left(A - c1_H\right)^k x, y \right\rangle \right. \tag{6.27}$$

$$- \frac{l_n + L_n}{2} \left[\frac{1}{(n+1)!} \left\langle A^{n+1} x, y \right\rangle - \frac{c^{n+1}}{(n+1)!} \langle x, y \rangle \right.$$

$$\left. \left. - \sum_{k=1}^{n} \frac{c^{n-k+1}}{k!\,(n-k+1)!} \left\langle \left(A - c1_H\right)^k x, y \right\rangle \right] \right|$$

$$\leq \frac{1}{2\,(n+1)!}\,(L_n - l_n)$$

$$\times \left[(c-m)^{n+1} \bigvee_{m}^{c} \left(\langle E_{(\cdot)} x, y\rangle\right) + (M-c)^{n+1} \bigvee_{c}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \right]$$

$$\leq \frac{1}{2\,(n+1)!}\,(L_n - l_n) \left(\frac{1}{2}\,(M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right)$$

$$\leq \frac{1}{2\,(n+1)!}\,(L_n - l_n) \left(\frac{1}{2}\,(M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \|x\|\,\|y\|$$

for any $x, y \in H$.

Proof. Consider the function $f : I \to \mathbb{R}$ defined by

$$f\left(t\right) := g\left(t\right) - \frac{1}{(n+1)!}\,\frac{L_n + l_n}{2} \cdot t^{n+1}.$$

Observe that

$$f^{(k)}\left(t\right) := g^{(k)}\left(t\right) - \frac{1}{(n-k+1)!}\,\frac{L_n + l_n}{2} \cdot t^{n-k+1}$$

for any $k = 0, ..., n$.

Since $g^{(n)}$ is (l_n, L_n)−Lipschitzian it follows that

$$f^{(n)}\left(t\right) := g^{(n)}\left(t\right) - \frac{L_n + l_n}{2} \cdot t$$

is $\frac{1}{2}\,(L_n - l_n)$-Lipschitzian and applying Theorem 371 for the function f, we deduce after required calculations the desired result (6.11). ∎

Remark 374 *In particular, we can state from (6.27) the following inequalities*

$$\left| \langle g(A) x, y \rangle - g(m) \langle x, y \rangle - \sum_{k=1}^{n} \frac{1}{k!} g^{(k)}(m) \left\langle (A - m1_H)^k x, y \right\rangle \right. \quad (6.28)$$

$$- \frac{l_n + L_n}{2} \left[\frac{1}{(n+1)!} \langle A^{n+1} x, y \rangle - \frac{m^{n+1}}{(n+1)!} \langle x, y \rangle \right.$$

$$\left. \left. - \sum_{k=1}^{n} \frac{m^{n-k+1}}{k!(n-k+1)!} \left\langle (A - m1_H)^k x, y \right\rangle \right] \right|$$

$$\leq \frac{1}{2(n+1)!} (L_n - l_n)(M - m)^{n+1} \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{2(n+1)!} (L_n - l_n)(M - m)^{n+1} \|x\| \|y\|$$

and

$$\left| \langle g(A) x, y \rangle - g(M) \langle x, y \rangle - \sum_{k=1}^{n} \frac{(-1)^k}{k!} g^{(k)}(M) \left\langle (M1_H - A)^k x, y \right\rangle \right.$$

$$- \frac{l_n + L_n}{2} \left[\frac{1}{(n+1)!} \langle A^{n+1} x, y \rangle - \frac{M^{n+1}}{(n+1)!} \langle x, y \rangle \right.$$

$$\left. \left. - \sum_{k=1}^{n} (-1)^k \frac{M^{n-k+1}}{k!(n-k+1)!} \left\langle (M1_H - A)^k x, y \right\rangle \right] \right|$$

$$\leq \frac{1}{2(n+1)!} (L_n - l_n)(M - m)^{n+1} \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{2(n+1)!} (L_n - l_n)(M - m)^{n+1} \|x\| \|y\| \quad (6.29)$$

and

$$\left| \langle g(A)x, y \rangle - g\left(\frac{m+M}{2}\right)\langle x, y \rangle \right. \tag{6.30}$$

$$-\sum_{k=1}^{n}\frac{1}{k!}g^{(k)}\left(\frac{m+M}{2}\right)\left\langle \left(A - \frac{m+M}{2}1_H\right)^k x, y \right\rangle$$

$$-\frac{l_n + L_n}{2}\left[\frac{1}{(n+1)!}\langle A^{n+1}x, y \rangle - \frac{1}{(n+1)!}\langle x, y \rangle\left(\frac{m+M}{2}\right)^{n+1}\right.$$

$$\left.\left. -\sum_{k=1}^{n}\frac{1}{(n-k+1)!k!}\left(\frac{m+M}{2}\right)^{n-k+1}\left\langle \left(A - \frac{m+M}{2}1_H\right)^k x, y \right\rangle\right]\right|$$

$$\leq \frac{1}{2^{n+2}(n+1)!}(L_n - l_n)(M-m)^{n+1}\bigvee_{m}^{M}(\langle E_{(\cdot)}x, y \rangle)$$

$$\leq \frac{1}{2^{n+2}(n+1)!}(L_n - l_n)(M-m)^{n+1}\|x\|\|y\|$$

respectively, for any $x, y \in H$.

6.2.3   Applications

By utilizing Theorem 369 and 371 for the exponential function, we can state the following result:

Proposition 375 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family, then for any $c \in [m, M]$ we have the inequality*

$$\left| \langle e^A x, y \rangle - e^c \sum_{k=0}^{n}\frac{1}{k!}\left\langle (A - c1_H)^k x, y \right\rangle \right| \tag{6.31}$$

$$\leq \frac{1}{n!}\left[(c-m)^n(e^c - e^m)\bigvee_{m}^{c}(\langle E_{(\cdot)}x, y \rangle)\right.$$

$$\left. + (M-c)^n(e^M - e^c)\bigvee_{c}^{M}(\langle E_{(\cdot)}x, y \rangle)\right]$$

$$\leq \frac{1}{n!}\max\left\{(M-c)^n(e^M - e^c), (c-m)^n(e^c - e^m)\right\}\bigvee_{m}^{M}(\langle E_{(\cdot)}x, y \rangle)$$

$$\leq \frac{1}{n!}\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^n(e^M - e^m)\bigvee_{m}^{M}(\langle E_{(\cdot)}x, y \rangle)$$

$$\leq \frac{1}{n!}\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^n(e^M - e^m)\|x\|\|y\|$$

and

$$\left| \left\langle e^A x, y \right\rangle - e^c \sum_{k=0}^{n} \frac{1}{k!} \left\langle (A - c1_H)^k x, y \right\rangle \right| \tag{6.32}$$

$$\leq \frac{1}{(n+1)!} e^M \left[(c-m)^{n+1} \bigvee_{m}^{c} \left(\left\langle E_{(\cdot)} x, y \right\rangle \right) + (M-c)^{n+1} \bigvee_{c}^{M} \left(\left\langle E_{(\cdot)} x, y \right\rangle \right) \right]$$

$$\leq \frac{1}{(n+1)!} e^M \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \bigvee_{m}^{M} \left(\left\langle E_{(\cdot)} x, y \right\rangle \right)$$

$$\leq \frac{1}{(n+1)!} e^M \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \|x\| \, \|y\|$$

for any $x, y \in H$.

Remark 376 *We observe that the best inequalities we can get from (6.31) and (6.32) are*

$$\left| \left\langle e^A x, y \right\rangle - e^{\frac{m+M}{2}} \sum_{k=0}^{n} \frac{1}{k!} \left\langle \left(A - \frac{m+M}{2} 1_H \right)^k x, y \right\rangle \right| \tag{6.33}$$

$$\leq \frac{1}{2^n n!} (M-m)^n \left(e^M - e^m \right) \bigvee_{m}^{M} \left(\left\langle E_{(\cdot)} x, y \right\rangle \right)$$

$$\leq \frac{1}{2^n n!} (M-m)^n \left(e^M - e^m \right) \|x\| \, \|y\|$$

and

$$\left| \left\langle e^A x, y \right\rangle - e^{\frac{m+M}{2}} \sum_{k=0}^{n} \frac{1}{k!} \left\langle \left(A - \frac{m+M}{2} 1_H \right)^k x, y \right\rangle \right| \tag{6.34}$$

$$\leq \frac{1}{2^{n+1}(n+1)!} e^M (M-m)^{n+1} \bigvee_{m}^{M} \left(\left\langle E_{(\cdot)} x, y \right\rangle \right)$$

$$\leq \frac{1}{2^{n+1}(n+1)!} e^M (M-m)^{n+1} \|x\| \, \|y\|$$

for any $x, y \in H$.

The same Theorems 369 and 371 applied for the logarithmic function produce:

Proposition 377 *Let A be a positive definite operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M] \subset (0, \infty)$ and $\{E_\lambda\}_\lambda$ be its spectral*

family, then for any $c \in [m, M]$ we have the inequalities

$$\left| \langle \ln Ax, y \rangle - \langle x, y \rangle \ln c - \sum_{k=1}^{n} \frac{(-1)^{k-1} \left\langle (A - c1_H)^k x, y \right\rangle}{kc^k} \right| \tag{6.35}$$

$$\leq \frac{1}{n} \left[\frac{(c-m)^n (c^n - m^n)}{c^n m^n} \bigvee_m^c \left(\langle E_{(\cdot)} x, y \rangle \right) \right.$$

$$\left. + \frac{(M-c)^n (M^n - c^n)}{M^m c^m} \bigvee_c^M \left(\langle E_{(\cdot)} x, y \rangle \right) \right]$$

$$\leq \frac{1}{n} \max \left\{ \frac{(c-m)^n (c^n - m^n)}{c^n m^n}, \frac{(M-c)^n (M^n - c^n)}{M^m c^m} \right\} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{n} \left(\frac{1}{2} (M-m) + \left| c - \frac{m+M}{2} \right| \right)^n \frac{(M^n - m^n)}{M^m m^m} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{n} \left(\frac{1}{2} (M-m) + \left| c - \frac{m+M}{2} \right| \right)^n \frac{(M^n - m^n)}{M^m m^m} \|x\| \, \|y\|$$

and

$$\left| \langle \ln Ax, y \rangle - \langle x, y \rangle \ln c - \sum_{k=1}^{n} \frac{(-1)^{k-1} \left\langle (A - c1_H)^k x, y \right\rangle}{kc^k} \right| \tag{6.36}$$

$$\leq \frac{1}{(n+1) m^{n+1}} \left[(c-m)^{n+1} \bigvee_m^c \left(\langle E_{(\cdot)} x, y \rangle \right) + (M-c)^{n+1} \bigvee_c^M \left(\langle E_{(\cdot)} x, y \rangle \right) \right]$$

$$\leq \frac{1}{(n+1) m^{n+1}} \left(\frac{1}{2} (M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{(n+1) m^{n+1}} \left(\frac{1}{2} (M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \|x\| \, \|y\|$$

for any $x, y \in H$.

Remark 378 *The best inequalities we can get from (6.35) and (6.36) are*

$$\left| \langle \ln Ax, y \rangle - \langle x, y \rangle \ln \left(\frac{m+M}{2} \right) - \sum_{k=1}^{n} \frac{(-1)^{k-1} \left\langle \left(A - \frac{m+M}{2} 1_H \right)^k x, y \right\rangle}{k \left(\frac{m+M}{2} \right)^k} \right|$$

$$\leq \frac{1}{2^n n} (M-m)^n \frac{(M^n - m^n)}{M^m m^m} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{2^n n} (M-m)^n \frac{(M^n - m^n)}{M^m m^m} \|x\| \, \|y\| \tag{6.37}$$

and

$$\left| \langle \ln Ax, y \rangle - \langle x, y \rangle \ln \left(\frac{m+M}{2} \right) - \sum_{k=1}^{n} \frac{(-1)^{k-1} \left\langle \left(A - \frac{m+M}{2} 1_H \right)^k x, y \right\rangle}{k \left(\frac{m+M}{2} \right)^k} \right|$$

$$\leq \frac{1}{2^{n+1} (n+1)} \left(\frac{M}{m} - 1 \right)^{n+1} \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{2^{n+1} (n+1)} \left(\frac{M}{m} - 1 \right)^{n+1} \|x\| \, \|y\| \tag{6.38}$$

for any $x, y \in H$.

6.3   Perturbed Version

6.3.1   Some Identities

The following result provides a perturbed Taylor's type representation for a function of selfadjoint operators in Hilbert spaces.

Theorem 379 (Dragomir, 2010, [4]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the equalities*

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (A - c 1_H)^k \tag{6.39}$$

$$+ \left[f(M) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (M - c)^k \right] 1_H$$

$$+ V_n(f, c, m, M)$$

where

$$V_n(f, c, m, M) \tag{6.40}$$

$$: \quad = \frac{(-1)^n}{(n-1)!} \int_{m-0}^{M} \left(\int_c^\lambda (t - \lambda)^{n-1} d \left(f^{(n)}(t) \right) \right) E_\lambda d\lambda.$$

Proof. We utilize the Taylor's formula for functions $f : I \to \mathbb{C}$ whose n-th derivative $f^{(n)}$ is locally of bounded variation on the interval I to write the equality

$$f(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (\lambda - c)^k + \frac{1}{n!} \int_c^\lambda (\lambda - t)^n d \left(f^{(n)}(t) \right) \tag{6.41}$$

for any $\lambda, c \in [m, M]$, where the integral is taken in the Riemann-Stieltjes sense.

If we integrate the equality on $[m, M]$ in the Riemann-Stieltjes sense with the integrator E_λ we get

$$\int_{m-0}^{M} f(\lambda) \, dE_\lambda = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \int_{m-0}^{M} (\lambda - c)^k \, dE_\lambda$$

$$+ \frac{1}{n!} \int_{m-0}^{M} \left(\int_{c}^{\lambda} (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right) dE_\lambda$$

which, by the spectral representation theorem, produces the equality

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (A - c 1_H)^k \qquad (6.42)$$

$$+ \frac{1}{n!} \int_{m-0}^{M} \left(\int_{c}^{\lambda} (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right) dE_\lambda$$

that is of interest in itself as well.

Now, integrating by parts in the Riemann-Stieltjes integral and using the Leibnitz formula for integrals with parameters, we have

$$\int_{m-0}^{M} \left(\int_{c}^{\lambda} (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right) dE_\lambda \qquad (6.43)$$

$$= E_\lambda \left(\int_{c}^{\lambda} (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right) \Bigg|_{m-0}^{M}$$

$$- \int_{m-0}^{M} E_\lambda d \left(\int_{c}^{\lambda} (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right)$$

$$= \left(\int_{c}^{M} (M - t)^n \, d\left(f^{(n)}(t) \right) \right) 1_H$$

$$- n \int_{m-0}^{M} \left(\int_{c}^{\lambda} (\lambda - t)^{n-1} \, d\left(f^{(n)}(t) \right) \right) E_\lambda d\lambda$$

and, since by the Taylor's formula (6.41) we have

$$\frac{1}{n!} \int_{c}^{M} (M - t)^n \, d\left(f^{(n)}(t) \right) = f(M) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (M - c)^k, \quad (6.44)$$

then, by (6.42) and (6.44), we deduce the equality (6.39) with the integral representation for the remainder provided by (6.40). ∎

The following particular instances are of interest for applications:

Corollary 380 (Dragomir, 2010, [4]) *With the assumptions of the above Theorem 379, we have the equalities*

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(m)(A - m1_H)^k \tag{6.45}$$

$$+ \left[f(M) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(m)(M - m)^k \right] 1_H + T_n(f, c, m, M)$$

where

$$T_n(f, m, M) \tag{6.46}$$

$$:= -\frac{1}{(n-1)!} \int_{m-0}^{M} \left(\int_{m}^{\lambda} (\lambda - t)^{n-1} d\left(f^{(n)}(t) \right) \right) E_\lambda d\lambda$$

and

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}\left(\frac{m+M}{2} \right) \left(A - \frac{m+M}{2} 1_H \right)^k \tag{6.47}$$

$$+ \left[f(M) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}\left(\frac{m+M}{2} \right) \left(\frac{M-m}{2} \right)^k \right] 1_H$$

$$+ W_n(f, c, m, M)$$

where

$$W_n(f, m, M) \tag{6.48}$$

$$:= \frac{(-1)^n}{(n-1)!} \int_{m-0}^{M} \left(\int_{\frac{m+M}{2}}^{\lambda} (t - \lambda)^{n-1} d\left(f^{(n)}(t) \right) \right) E_\lambda d\lambda$$

and

$$f(A) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(M)(M1_H - A)^k + Y_n(f, c, m, M) \tag{6.49}$$

where

$$Y_n(f, m, M) \tag{6.50}$$

$$:= \frac{(-1)^{n+1}}{(n-1)!} \int_{m-0}^{M} \left(\int_{\lambda}^{M} (t - \lambda)^{n-1} d\left(f^{(n)}(t) \right) \right) E_\lambda d\lambda,$$

respectively.

Remark 381 *In order to give some examples we use the simplest representation, namely (6.49) for the exponential and the logarithmic functions.*

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then we have the representation

$$e^A = e^M \sum_{k=0}^{n} \frac{(-1)^k}{k!} (M1_H - A)^k \tag{6.51}$$

$$+ \frac{(-1)^{n+1}}{(n-1)!} \int_{m-0}^{M} \left(\int_{\lambda}^{M} (t - \lambda)^{n-1} e^t dt \right) E_\lambda d\lambda.$$

In the case when A is positive definite, i.e., $m > 0$, then we have the representation

$$\ln A = (\ln M) 1_H - \sum_{k=1}^{n} \frac{(M1_H - A)^k}{kM^k} \tag{6.52}$$

$$- n \int_{m-0}^{M} \left(\int_{\lambda}^{M} \frac{(t - \lambda)^{n-1}}{t^{n+1}} dt \right) E_\lambda d\lambda.$$

6.3.2   Error Bounds for $f^{(n)}$ of Bounded Variation

We start with the following result that provides an approximation for an n-time differentiable function of selfadjoint operators in Hilbert spaces:

Theorem 382 (Dragomir, 2010, [4]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on*

the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequalities

$$|\langle V_n\left(f, c, m, M\right) x, y\rangle| \tag{6.53}$$

$$\leq \frac{1}{(n-1)!} \int_{m-0}^{c} (c-\lambda)^{n-1} \bigvee_{\lambda}^{c}\left(f^{(n)}\right) |\langle E_\lambda x, y\rangle|\, d\lambda$$

$$+ \frac{1}{(n-1)!} \int_{c}^{M} (\lambda-c)^{n-1} \bigvee_{c}^{\lambda}\left(f^{(n)}\right) |\langle E_\lambda x, y\rangle|\, d\lambda$$

$$\leq \frac{1}{(n-1)!} \bigvee_{m}^{c}\left(f^{(n)}\right) \int_{m-0}^{c} (c-\lambda)^{n-1} |\langle E_\lambda x, y\rangle|\, d\lambda$$

$$+ \frac{1}{(n-1)!} \bigvee_{c}^{M}\left(f^{(n)}\right) \int_{c}^{M} (\lambda-c)^{n-1} |\langle E_\lambda x, y\rangle|\, d\lambda$$

$$\leq \frac{1}{(n-1)!} \max\left\{\bigvee_{m}^{c}\left(f^{(n)}\right), \bigvee_{c}^{M}\left(f^{(n)}\right)\right\} \int_{m-0}^{M} |\lambda-c|^{n-1} |\langle E_\lambda x, y\rangle|\, d\lambda$$

$$\leq \frac{1}{n!} \max\left\{\bigvee_{m}^{c}\left(f^{(n)}\right), \bigvee_{c}^{M}\left(f^{(n)}\right)\right\} B_n(c, m, M, x, y),$$

for any $x, y \in H$, where

$$B_n(c, m, M, x, y) \tag{6.54}$$

$$:= \begin{cases} \left[(M-c)^n + (c-m)^n\right] \|x\|\, \|y\|\,; \\[2mm] C_n(c, m, M, x, y); \\[2mm] n \left[\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right]^{n-1} \\ \quad \times \left[\langle(M1_H - A)x, x\rangle \langle(M1_H - A)y, y\rangle\right]^{1/2} \end{cases}$$

and

$$C_n(c, m, M, x, y) \tag{6.55}$$
$$:= \left[\langle[(M-c)^n 1_H - sgn\left(A - c1_H\right)|A - c1_H|^n]x, x\rangle\right]^{1/2}$$
$$\times \left[\langle[(M-c)^n 1_H - sgn\left(A - c1_H\right)|A - c1_H|^n]y, y\rangle\right]^{1/2}.$$

Here the operator function $sgn\left(A - c1_H\right)|A - c1_H|^n$ is generated by the continuous function $sgn\left(\cdot - c\right)|\cdot - c|^n$ defined on the interval $[m, M]$.

Proof. From the identities (6.39) and (6.40) we have

$$
|\langle V_n \left(f, c, m, M \right) x, y \rangle| \tag{6.56}
$$

$$
= \left| \frac{1}{(n-1)!} \int_{m-0}^{M} \left(\int_{c}^{\lambda} (t-\lambda)^{n-1} d \left(f^{(n)}(t) \right) \right) \langle E_\lambda x, y \rangle \, d\lambda \right|
$$

$$
\leq \frac{1}{(n-1)!} \left| \int_{m-0}^{c} \left(\int_{c}^{\lambda} (t-\lambda)^{n-1} d \left(f^{(n)}(t) \right) \right) \langle E_\lambda x, y \rangle \, d\lambda \right|
$$

$$
+ \frac{1}{(n-1)!} \left| \int_{c}^{M} \left(\int_{c}^{\lambda} (t-\lambda)^{n-1} d \left(f^{(n)}(t) \right) \right) \langle E_\lambda x, y \rangle \, d\lambda \right|
$$

$$
\leq \frac{1}{(n-1)!} \int_{m-0}^{c} \left| \int_{c}^{\lambda} (t-\lambda)^{n-1} d \left(f^{(n)}(t) \right) \right| |\langle E_\lambda x, y \rangle| \, d\lambda
$$

$$
+ \frac{1}{(n-1)!} \int_{c}^{M} \left| \int_{c}^{\lambda} (t-\lambda)^{n-1} d \left(f^{(n)}(t) \right) \right| |\langle E_\lambda x, y \rangle| \, d\lambda
$$

for any $x, y \in H$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$
\left| \int_a^b p(t) \, dv(t) \right| \leq \sup_{t \in [a,b]} |p(t)| \bigvee_a^b (v), \tag{6.57}
$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

By the same property (6.57) we have

$$
\left| \int_c^\lambda (t-\lambda)^{n-1} d \left(f^{(n)}(t) \right) \right| \leq (c-\lambda)^{n-1} \bigvee_\lambda^c \left(f^{(n)} \right) \tag{6.58}
$$

for $\lambda \in [m, c]$ and

$$
\left| \int_c^\lambda (t-\lambda)^{n-1} d \left(f^{(n)}(t) \right) \right| \leq (\lambda-c)^{n-1} \bigvee_c^\lambda \left(f^{(n)} \right) \tag{6.59}
$$

for $\lambda \in [c, M]$.

Now, on making use of (6.56) and (6.58)-(6.59) we deduce

$$|\langle V_n\left(f,c,m,M\right)x,y\rangle|$$

$$\leq \frac{1}{(n-1)!}\int_{m-0}^{c}(c-\lambda)^{n-1}\bigvee_{\lambda}^{c}\left(f^{(n)}\right)|\langle E_\lambda x,y\rangle|\,d\lambda$$

$$+\frac{1}{(n-1)!}\int_{c}^{M}(\lambda-c)^{n-1}\bigvee_{c}^{\lambda}\left(f^{(n)}\right)|\langle E_\lambda x,y\rangle|\,d\lambda$$

for any $x,y \in H$ which proves the first part of (6.53).

The second and the third inequalities follow by the properties of the integral.

For the last part we observe that

$$\int_{m-0}^{M}|\lambda-c|^{n-1}|\langle E_\lambda x,y\rangle|\,d\lambda \leq \max_{\lambda\in[m,M]}|\langle E_\lambda x,y\rangle|\int_{m}^{M}|\lambda-c|^{n-1}\,d\lambda$$

$$\leq \frac{1}{n}\,\|x\|\,\|y\|\,[(M-c)^{n}+(c-m)^{n}]$$

for any $x,y \in H$, and the proof for the first branch of $B(c,m,M,x,y)$ is complete.

Now, to prove the inequality for the second branch of $B(c,m,M,x,y)$ we use the fact that if P is a nonnegative operator on H, i.e., $\langle Px,x\rangle \geq 0$ for any $x \in H$, then the following inequality that provides a generalization of the Schwarz inequality in H can be stated

$$|\langle Px,y\rangle|^{2} \leq \langle Px,x\rangle\langle Py,y\rangle \tag{6.60}$$

for any $x,y \in H$.

If we use (6.60) and the Cauchy-Buniakowski-Schwarz weighted integral inequality we can write that

$$\int_{m-0}^{M}|\lambda-c|^{n-1}|\langle E_\lambda x,y\rangle|\,d\lambda \tag{6.61}$$

$$\leq \int_{m-0}^{M}|\lambda-c|^{n-1}\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\,d\lambda$$

$$\leq \left(\int_{m-0}^{M}|\lambda-c|^{n-1}\langle E_\lambda x,x\rangle\,d\lambda\right)^{1/2}\left(\int_{m-0}^{M}|\lambda-c|^{n-1}\langle E_\lambda y,y\rangle\,d\lambda\right)^{1/2}$$

for any $x,y \in H$.

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x \rangle \, d\lambda \tag{6.62}$$

$$= \int_{m-0}^{c} (c - \lambda)^{n-1} \langle E_\lambda x, x \rangle \, d\lambda + \int_{c}^{M} (\lambda - c)^{n-1} \langle E_\lambda x, x \rangle \, d\lambda$$

$$= \frac{1}{n} \left[-\int_{m-0}^{c} \langle E_\lambda x, x \rangle \, d (c - \lambda)^n + \int_{c}^{M} \langle E_\lambda x, x \rangle \, d (\lambda - c)^n \right]$$

$$= \frac{1}{n} \left[-(c - \lambda)^n \langle E_\lambda x, x \rangle |_{m-0}^{c} + \int_{m-0}^{c} (c - \lambda)^n \, d \langle E_\lambda x, x \rangle \right]$$

$$+ \frac{1}{n} \left[\langle E_\lambda x, x \rangle (\lambda - c)^n |_{c}^{M} - \int_{c}^{M} (\lambda - c)^n \, d \langle E_\lambda x, x \rangle \right]$$

$$= \frac{1}{n} \int_{m-0}^{c} (c - \lambda)^n \, d \langle E_\lambda x, x \rangle$$

$$+ \frac{1}{n} \left[\|x\|^2 (M - c)^n - \int_{c}^{M} (\lambda - c)^n \, d \langle E_\lambda x, x \rangle \right]$$

$$= \frac{1}{n} \|x\|^2 (M - c)^n$$

$$+ \frac{1}{n} \left[\int_{m-0}^{c} (c - \lambda)^n \, d \langle E_\lambda x, x \rangle - \int_{c}^{M} (\lambda - c)^n \, d \langle E_\lambda x, x \rangle \right]$$

$$= \frac{1}{n} \left[\|x\|^2 (M - c)^n - \int_{m-0}^{M} sgn (\lambda - c) |\lambda - c|^n \, d \langle E_\lambda x, x \rangle \right]$$

$$= \frac{1}{n} [\langle [(M - c)^n 1_H - sgn (A - c1_H) |A - c1_H|^n] x, x \rangle]$$

for any $x \in H$, and a similar relation for y, namely

$$\int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda y, y \rangle \, d\lambda \tag{6.63}$$

$$= \frac{1}{n} [\langle [(M - c)^n 1_H - sgn (A - c1_H) |A - c1_H|^n] y, y \rangle]$$

for any $y \in H$.

The inequality (6.61) and the equalities (6.62) and (6.63) produce the second bound in (6.54).

Finally, observe also that

$$\int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x \rangle \, d\lambda \tag{6.64}$$

$$= \int_{m-0}^{c} (c - \lambda)^{n-1} \langle E_\lambda x, x \rangle \, d\lambda + \int_{c}^{M} (\lambda - c)^{n-1} \langle E_\lambda x, x \rangle \, d\lambda$$

$$\leq (c - m)^{n-1} \int_{m-0}^{c} \langle E_\lambda x, x \rangle \, d\lambda + (M - c)^{n-1} \int_{c}^{M} \langle E_\lambda x, x \rangle \, d\lambda$$

$$\leq \max \left\{ (c - m)^{n-1}, (M - c)^{n-1} \right\} \int_{m-0}^{M} \langle E_\lambda x, x \rangle \, d\lambda$$

$$= \left[\frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right]^{n-1}$$

$$\times \left[\langle E_\lambda x, x \rangle \lambda |_{m-0}^{M} - \int_{m-0}^{M} \lambda d \langle E_\lambda x, x \rangle \right]$$

$$= \left[\frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right]^{n-1} \langle (M 1_H - A) x, x \rangle$$

for any $x \in H$ and similarly,

$$\int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x \rangle \, d\lambda \tag{6.65}$$

$$\leq \left[\frac{1}{2} (M - m) + \left| c - \frac{m + M}{2} \right| \right]^{n-1} \langle (M 1_H - A) y, y \rangle$$

for any $y \in H$.

On making use of (6.61), (6.64) and (6.65) we deduce the last bound provided in (6.54). ∎

The following particular cases are of interest for applications

Corollary 383 (Dragomir, 2010, [4]) *With the assumption of Theorem 382 we have the inequalities*

$$|\langle T_n (f, m, M) x, y \rangle| \tag{6.66}$$

$$\leq \frac{1}{(n-1)!} \int_{m-0}^{M} (\lambda - m)^{n-1} \bigvee_{m}^{\lambda} \left(f^{(n)} \right) |\langle E_\lambda x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{(n-1)!} \bigvee_{m}^{M} \left(f^{(n)} \right) \int_{m-0}^{M} (\lambda - m)^{n-1} |\langle E_\lambda x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{n!} \bigvee_{m}^{M} \left(f^{(n)} \right) B_n(m, M, x, y),$$

for any $x, y \in H$, where

$$B_n(m, M, x, y) \tag{6.67}$$

$$:= \begin{cases} (M - m)^n \, \|x\| \, \|y\| \, ; \\[2ex] C_n(m, M, x, y); \\[2ex] n \, (M - m)^{n-1} \left[\langle (M1_H - A) \, x, x \rangle \, \langle (M1_H - A) \, y, y \rangle \right]^{1/2} \end{cases}$$

and

$$C_n(m, M, x, y) \tag{6.68}$$
$$:= \left[\langle \left[(M - m)^n \, 1_H - (A - m1_H)^n \right] x, x \rangle \right]^{1/2}$$
$$\times \left[\langle \left[(M - m)^n \, 1_H - (A - m1_H)^n \right] y, y \rangle \right]^{1/2}.$$

The proof follows from Theorem 382 by choosing $c = m$ and performing the corresponding calculations.

Corollary 384 (Dragomir, 2010, [4]) *With the assumption of Theorem 382 we have the inequalities*

$$|\langle Y_n(f, m, M) \, x, y \rangle| \tag{6.69}$$

$$\leq \frac{1}{(n-1)!} \int_{m-0}^{M} (M - \lambda)^{n-1} \bigvee_{\lambda}^{M} \left(f^{(n)} \right) |\langle E_\lambda x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{(n-1)!} \bigvee_{m}^{M} \left(f^{(n)} \right) \int_{m-0}^{M} (M - \lambda)^{n-1} |\langle E_\lambda x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{n!} \bigvee_{m}^{M} \left(f^{(n)} \right) \tilde{B}_n(m, M, x, y),$$

for any $x, y \in H$, where

$$\tilde{B}_n(m, M, x, y) \tag{6.70}$$

$$:= \begin{cases} (M - m)^n \, \|x\| \, \|y\| \, ; \\[2ex] \tilde{C}_n(m, M, x, y); \\[2ex] n \, (M - m)^{n-1} \left[\langle (M1_H - A) \, x, x \rangle \, \langle (M1_H - A) \, y, y \rangle \right]^{1/2} \end{cases}$$

and

$$\tilde{C}_n(m, M, x, y) \tag{6.71}$$
$$:= \left[\langle (M1_H - A)^n \, x, x \rangle \right]^{1/2} \left[\langle (M1_H - A)^n \, y, y \rangle \right]^{1/2}.$$

The proof follows from Theorem 382 by choosing $c = M$ and performing the corresponding calculations.

The best bound we can get is incorporated in

Corollary 385 (Dragomir, 2010, [4]) *With the assumption of Theorem 382 we have the inequalities*

$$|\langle W_n(f, m, M) x, y\rangle| \tag{6.72}$$

$$\leq \frac{1}{(n-1)!} \int_{m-0}^{\frac{m+M}{2}} \left(\frac{m+M}{2} - \lambda\right)^{n-1} \bigvee_{\lambda}^{\frac{m+M}{2}} \left(f^{(n)}\right) |\langle E_\lambda x, y\rangle| \, d\lambda$$

$$+ \frac{1}{(n-1)!} \int_{\frac{m+M}{2}}^{M} \left(\lambda - \frac{m+M}{2}\right)^{n-1} \bigvee_{\frac{m+M}{2}}^{\lambda} \left(f^{(n)}\right) |\langle E_\lambda x, y\rangle| \, d\lambda$$

$$\leq \frac{1}{(n-1)!} \bigvee_{m}^{\frac{m+M}{2}} \left(f^{(n)}\right) \int_{m-0}^{\frac{m+M}{2}} \left(\frac{m+M}{2} - \lambda\right)^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda$$

$$+ \frac{1}{(n-1)!} \bigvee_{\frac{m+M}{2}}^{M} \left(f^{(n)}\right) \int_{\frac{m+M}{2}}^{M} \left(\lambda - \frac{m+M}{2}\right)^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda$$

$$\leq \frac{1}{(n-1)!} \max\left\{ \bigvee_{m}^{\frac{m+M}{2}} \left(f^{(n)}\right), \bigvee_{\frac{m+M}{2}}^{M} \left(f^{(n)}\right) \right\}$$

$$\times \int_{m-0}^{M} \left|\lambda - \frac{m+M}{2}\right|^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda$$

$$\leq \frac{1}{n!} \max\left\{ \bigvee_{m}^{\frac{m+M}{2}} \left(f^{(n)}\right), \bigvee_{\frac{m+M}{2}}^{M} \left(f^{(n)}\right) \right\} \check{B}_n(m, M, x, y),$$

for any $x, y \in H$, where

$$\check{B}_n(m, M, x, y) \tag{6.73}$$

$$: = \begin{cases} \frac{(M-m)^n}{2^{n-1}} \|x\| \|y\|; \\ \\ \check{C}(m, M, x, y) \\ \\ \frac{n}{2^{n-1}} (M - m)^{n-1} \left[\langle (M1_H - A) x, x\rangle \langle (M1_H - A) y, y\rangle\right]^{1/2} \end{cases}$$

and

$$\check{C}_n(m, M, x, y) \tag{6.74}$$

$$:= \left[\left\langle \left[\frac{(M-m)^n}{2^n} 1_H - sgn\left(A - \frac{m+M}{2} 1_H \right) \left| A - \frac{m+M}{2} 1_H \right|^n \right] x, x \right\rangle \right]^{1/2}$$

$$\times \left[\left\langle \left[\frac{(M-m)^n}{2^n} 1_H - sgn\left(A - \frac{m+M}{2} 1_H \right) \left| A - \frac{m+M}{2} 1_H \right|^n \right] y, y \right\rangle \right]^{1/2}.$$

6.3.3   Error Bounds for $f^{(n)}$ Lipschitzian

The case when the n-th derivative is Lipschitzian is incorporated in the following result:

Theorem 386 (Dragomir, 2010, [4]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \rightarrow \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequalities*

$$|\langle V_n(f, c, m, M) x, y \rangle| \tag{6.75}$$

$$\leq \frac{1}{n!} L_n \int_{m-0}^{M} |\lambda - c|^n |\langle E_\lambda x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{(n+1)!} L_n$$

$$\times \begin{cases} \left[(M-c)^{n+1} + (c-m)^{n+1} \right] \|x\| \|y\|; \\ \\ \left[\left\langle \left[(M-c)^{n+1} 1_H - sgn(A - c1_H) |A - c1_H|^{n+1} \right] x, x \right\rangle \right]^{1/2} \\ \quad \times \left[\left\langle \left[(M-c)^{n+1} 1_H - sgn(A - c1_H) |A - c1_H|^{n+1} \right] y, y \right\rangle \right]^{1/2}; \\ \\ (n+1) \left[\frac{1}{2} (M-m) + \left| c - \frac{m+M}{2} \right| \right]^n \\ \quad \times \left[\langle (M1_H - A) x, x \rangle \langle (M1_H - A) y, y \rangle \right]^{1/2}; \end{cases}$$

for any $x, y \in H$.

Proof. From the inequality (6.56) in the proof of Theorem 382 we have

$$|\langle V_n\left(f,c,m,M\right)x,y\rangle| \tag{6.76}$$

$$\leq \frac{1}{(n-1)!}\int_{m-0}^{c}\left|\int_{\lambda}^{c}(t-\lambda)^{n-1}\,d\left(f^{(n)}\left(t\right)\right)\right||\langle E_\lambda x,y\rangle|\,d\lambda$$

$$+\frac{1}{(n-1)!}\int_{c}^{M}\left|\int_{c}^{\lambda}(t-\lambda)^{n-1}\,d\left(f^{(n)}\left(t\right)\right)\right||\langle E_\lambda x,y\rangle|\,d\lambda$$

for any $x,y \in H$.

Further, we utilize the fact that for an $L-$Lipschitzian function, p : $[\alpha,\beta]\to\mathbb{C}$ and a Riemann integrable function $v:[\alpha,\beta]\to\mathbb{C}$, the Riemann-Stieltjes integral $\int_\alpha^\beta p\left(s\right)dv\left(s\right)$ exists and

$$\left|\int_\alpha^\beta p\left(s\right)dv\left(s\right)\right|\leq L\int_\alpha^\beta|p\left(s\right)|\,ds.$$

On making use of this property we have for $\lambda\in[m,c]$ that

$$\left|\int_\lambda^c(t-\lambda)^{n-1}\,d\left(f^{(n)}\left(t\right)\right)\right|\leq L_n\int_\lambda^c(t-\lambda)^{n-1}\,dt=\frac{1}{n}L_n\left(c-\lambda\right)^n$$

and for $\lambda\in[c,M]$ that

$$\left|\int_c^\lambda(t-\lambda)^{n-1}\,d\left(f^{(n)}\left(t\right)\right)\right|\leq L_n\int_c^\lambda(\lambda-t)^{n-1}\,dt=\frac{1}{n}L_n\left(\lambda-c\right)^n$$

which, by (6.76) produces the inequality

$$|\langle V_n\left(f,c,m,M\right)x,y\rangle| \tag{6.77}$$

$$\leq\frac{1}{n!}L_n\int_{m-0}^c(c-\lambda)^n\,|\langle E_\lambda x,y\rangle|\,d\lambda+\frac{1}{n!}L_n\int_c^M(\lambda-c)^n\,|\langle E_\lambda x,y\rangle|\,d\lambda$$

$$=\frac{1}{n!}L_n\int_{m-0}^M|\lambda-c|^n\,|\langle E_\lambda x,y\rangle|\,d\lambda,$$

for any $x,y \in H$, and the first part of (6.75) is proved.

Finally, we observe that the bounds for the integral $\int_{m-0}^M|\lambda-c|^n\,|\langle E_\lambda x,y\rangle|\,d\lambda$ can be obtained in a similar manner to the ones from the proof of Theorem 382 and the details are omitted. ∎

The following result contains error bounds for the particular expansions considered in Corollary 380:

Corollary 387 (Dragomir, 2010, [4]) *With the assumptions in Theorem 386 we have the inequalities*

$$|\langle T_n(f,m,M)x,y\rangle| \tag{6.78}$$

$$\leq \frac{1}{n!}L_n \int_{m-0}^{M} (\lambda-m)^n |\langle E_\lambda x,y\rangle|\, d\lambda$$

$$\leq \frac{1}{(n+1)!}L_n$$

$$\times \begin{cases} (M-m)^{n+1}\,\|x\|\,\|y\|\,; \\[2mm] \left[\left\langle\left[(M-m)^{n+1}1_H-(A-m1_H)^{n+1}\right]x,x\right\rangle\right]^{1/2} \\ \quad\times\left[\left\langle\left[(M-m)^{n+1}1_H-(A-m1_H)^{n+1}\right]y,y\right\rangle\right]^{1/2}\,; \\[2mm] (n+1)(M-m)^n\left[\langle(M1_H-A)x,x\rangle\langle(M1_H-A)y,y\rangle\right]^{1/2}\,; \end{cases}$$

and

$$|\langle Y_n(f,m,M)x,y\rangle| \tag{6.79}$$

$$\leq \frac{1}{n!}L_n \int_{m-0}^{M} (M-\lambda)^n |\langle E_\lambda x,y\rangle|\, d\lambda$$

$$\leq \frac{1}{(n+1)!}L_n$$

$$\times \begin{cases} (M-m)^{n+1}\,\|x\|\,\|y\|\,; \\[2mm] \left[\left\langle\left[(M1_H-A)^{n+1}\right]x,x\right\rangle\right]^{1/2}\left[\left\langle\left[(M1_H-A)^{n+1}\right]y,y\right\rangle\right]^{1/2}\,; \\[2mm] (n+1)\left[(M-m)\right]^n\left[\langle(M1_H-A)x,x\rangle\langle(M1_H-A)y,y\rangle\right]^{1/2}\,; \end{cases}$$

and

$$|\langle W_n(f,m,M)x,y\rangle| \tag{6.80}$$

$$\leq \frac{1}{n!}L_n \int_{m-0}^{M} \left|\lambda-\frac{m+M}{2}\right|^n |\langle E_\lambda x,y\rangle|\, d\lambda \leq \frac{1}{(n+1)!}L_n$$

$$\times \begin{cases} \frac{(M-m)^{n+1}}{2^n}\,\|x\|\,\|y\|\,; \\[2mm] \left[\left\langle\left[\frac{(M-m)^{n+1}}{2^n}1_H-sgn\left(A-\frac{m+M}{2}1_H\right)\left|A-\frac{m+M}{2}1_H\right|^{n+1}\right]x,x\right\rangle\right]^{1/2} \\ \quad\times\left[\left\langle\left[\frac{(M-m)^{n+1}}{2^n}1_H-sgn\left(A-\frac{m+M}{2}1_H\right)\left|A-\frac{m+M}{2}1_H\right|^{n+1}\right]y,y\right\rangle\right]^{1/2}\,; \\[2mm] \frac{n+1}{2^n}(M-m)^n\left[\langle(M1_H-A)x,x\rangle\langle(M1_H-A)y,y\rangle\right]^{1/2}\,; \end{cases}$$

for any $x,y\in H$, *respectively.*

6.3.4   Applications

In order to obtain various vectorial operator inequalities one can use the above results for particular elementary functions. We restrict ourself to only two examples of functions, namely the exponential and the logarithmic functions.

If we apply Corollary 384 for the exponential function, we can state the following result:

Proposition 388 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family. Then we have*

$$\left| \left\langle e^A x, y \right\rangle - e^M \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left\langle (M1_H - A)^k x, y \right\rangle \right| \tag{6.81}$$

$$\leq \frac{1}{(n-1)!} \int_{m-0}^{M} (M - \lambda)^{n-1} \left(e^M - e^\lambda \right) |\langle E_\lambda x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{(n-1)!} \left(e^M - e^m \right) \int_{m-0}^{M} (M - \lambda)^{n-1} |\langle E_\lambda x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{n!} \left(e^M - e^m \right)$$

$$\times \begin{cases} (M - m)^n \, \|x\| \, \|y\| \, ; \\[2mm] \left[\langle (M1_H - A)^n x, x \rangle \right]^{1/2} \left[\langle (M1_H - A)^n y, y \rangle \right]^{1/2} \\[2mm] n \, (M - m)^{n-1} \left[\langle (M1_H - A) x, x \rangle \langle (M1_H - A) y, y \rangle \right]^{1/2} \end{cases}$$

for any $x, y \in H$.

If we use Corollary 387 then we can provide other bounds as follows:

Proposition 389 *With the assumptions of Proposition 388 we have*

$$
\left| \langle e^A x, y \rangle - e^M \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left\langle (M1_H - A)^k x, y \right\rangle \right| \tag{6.82}
$$

$$
\leq \frac{1}{n!} e^M \int_{m-0}^{M} (M - \lambda)^n \left| \langle E_\lambda x, y \rangle \right| d\lambda
$$

$$
\leq \frac{1}{(n+1)!} e^M
$$

$$
\times
\begin{cases}
(M - m)^{n+1} \|x\| \|y\| ; \\[2mm]
\left[\left\langle \left[(M1_H - A)^{n+1} \right] x, x \right\rangle \right]^{1/2} \left[\left\langle \left[(M1_H - A)^{n+1} \right] y, y \right\rangle \right]^{1/2} ; \\[2mm]
(n+1) \left[(M - m) \right]^n \left[\langle (M1_H - A) x, x \rangle \langle (M1_H - A) y, y \rangle \right]^{1/2} ;
\end{cases}
$$

Finally, the Corollaries 384 and 387 produce the following results for the logarithmic function:

Proposition 390 *Let A be a positive definite operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M] \subset (0, \infty)$ and $\{E_\lambda\}_\lambda$ be its spectral family, then*

$$
\left| \langle \ln A x, y \rangle - \langle x, y \rangle \ln M + \sum_{k=1}^{n} \frac{\left\langle (M1_H - A)^k x, y \right\rangle}{k M^k} \right| \tag{6.83}
$$

$$
\leq \int_{m-0}^{M} (M - \lambda)^{n-1} \frac{M^n - \lambda^n}{M^n \lambda^n} \left| \langle E_\lambda x, y \rangle \right| d\lambda
$$

$$
\leq \frac{M^n - m^n}{M^n m^n} \int_{m-0}^{M} (M - \lambda)^{n-1} \left| \langle E_\lambda x, y \rangle \right| d\lambda
$$

$$
\leq \frac{M^n - m^n}{n M^n m^n}
$$

$$
\times
\begin{cases}
(M - m)^n \|x\| \|y\| ; \\[2mm]
\left[\langle (M1_H - A)^n x, x \rangle \right]^{1/2} \left[\langle (M1_H - A)^n y, y \rangle \right]^{1/2} \\[2mm]
n (M - m)^{n-1} \left[\langle (M1_H - A) x, x \rangle \langle (M1_H - A) y, y \rangle \right]^{1/2}
\end{cases}
$$

and

$$\left| \langle \ln Ax, y \rangle - \langle x, y \rangle \ln M + \sum_{k=1}^{n} \frac{\langle (M1_H - A)^k x, y \rangle}{kM^k} \right| \tag{6.84}$$

$$\leq \frac{1}{m^{n+1}} \int_{m-0}^{M} (M - \lambda)^n |\langle E_\lambda x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{(n+1) \, m^{n+1}}$$

$$\times \begin{cases} (M - m)^{n+1} \, \|x\| \, \|y\| \, ; \\[2mm] \left[\langle (M1_H - A)^{n+1} x, x \rangle \right]^{1/2} \left[\langle (M1_H - A)^{n+1} y, y \rangle \right]^{1/2} ; \\[2mm] (n+1) \left[(M - m) \right]^n \left[\langle (M1_H - A) x, x \rangle \langle (M1_H - A) y, y \rangle \right]^{1/2} ; \end{cases}$$

6.4   Two Points Taylor's Type Inequalities

6.4.1   *Representation Results*

We start with the following identity that has been obtained in [2]. For the sake of completeness we give here a short proof as well.

Lemma 391 (Dragomir, 2010, [2]) *Let I be a closed subinterval on $\mathbb{R}$, let $a, b \in I$ with $a < b$ and let n be a nonnegative integer. If $f : I \to \mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[a, b]$, then, for any $x \in [a, b]$ we have the representation*

$$f(x) = \frac{1}{b-a} \left[(b-x) f(a) + (x-a) f(b) \right] \tag{6.85}$$

$$+ \frac{(b-x)(x-a)}{b-a}$$

$$\times \sum_{k=1}^{n} \frac{1}{k!} \left\{ (x-a)^{k-1} f^{(k)}(a) + (-1)^k (b-x)^{k-1} f^{(k)}(b) \right\}$$

$$+ \frac{1}{b-a} \int_a^b S_n(x, t) \, d\left(f^{(n)}(t) \right),$$

where the kernel $S_n : [a, b]^2 \to \mathbb{R}$ is given by

$$S_n(x, t) = \frac{1}{n!} \times \begin{cases} (x-t)^n (b-x) & \text{if } a \leq t \leq x; \\[2mm] (-1)^{n+1} (t-x)^n (x-a) & \text{if } x < t \leq b \end{cases} \tag{6.86}$$

and the integral in the remainder is taken in the Riemann-Stieltjes sense.

Proof. We utilize the following Taylor's representation formula for functions $f : I \to \mathbb{R}$ such that the n-th derivatives $f^{(n)}$ are of locally bounded variation on the interval I,

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} (x-c)^k f^{(k)}(c) + \frac{1}{n!} \int_c^x (x-t)^n \, d\left(f^{(n)}(t)\right), \qquad (6.87)$$

where x and c are in I and the integral in the remainder is taken in the Riemann-Stieltjes sense.

Choosing $c = a$ and then $c = b$ in (6.87) we can write that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} (x-a)^k f^{(k)}(a) + \frac{1}{n!} \int_a^x (x-t)^n \, d\left(f^{(n)}(t)\right), \qquad (6.88)$$

and

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} (b-x)^k f^{(k)}(b) \qquad (6.89)$$
$$+ \frac{(-1)^{n+1}}{n!} \int_x^b (t-x)^n \, d\left(f^{(n)}(t)\right),$$

for any $x \in [a,b]$.

Now, by multiplying (6.88) with $(b-x)$ and (6.89) with $(x-a)$ we get

$$(b-x) f(x) \qquad (6.90)$$
$$= (b-x) f(a) + (b-x)(x-a) \sum_{k=1}^{n} \frac{1}{k!} (x-a)^{k-1} f^{(k)}(a)$$
$$+ \frac{1}{n!} (b-x) \int_a^x (x-t)^n \, d\left(f^{(n)}(t)\right)$$

and

$$(x-a) f(x) \qquad (6.91)$$
$$= (x-a) f(b) + (b-x)(x-a) \sum_{k=1}^{n} \frac{(-1)^k}{k!} (b-x)^{k-1} f^{(k)}(b)$$
$$+ \frac{(-1)^{n+1}}{n!} (x-a) \int_x^b (t-x)^n \, d\left(f^{(n)}(t)\right)$$

respectively.

Finally, by adding the equalities (6.90) and (6.91) and dividing the sum with $(b-a)$, we obtain the desired representation (6.86). ∎

Remark 392 *The case $n = 0$ provides the representation*

$$f(x) = \frac{1}{b-a}\left[(b-x)f(a) + (x-a)f(b)\right] \tag{6.92}$$

$$+ \frac{1}{b-a}\int_a^b S(x,t)\, d(f(t))$$

for any $x \in [a, b]$, where

$$S(x,t) = \begin{cases} b - x & \text{if } a \le t \le x, \\ a - x & \text{if } x < t \le b, \end{cases}$$

and f is of bounded variation on $[a, b]$. This result was obtained by a different approach in [1].

The case $n = 1$ provides the representation

$$f(x) = \frac{1}{b-a}\left[(b-x)f(a) + (x-a)f(b)\right] \tag{6.93}$$

$$+ \frac{1}{b-a}\int_a^b Q(x,t)\, d(f'(t)),$$

where

$$Q(x,t) = \begin{cases} (a - t)(b - x) & \text{if } a \le t \le x, \\ (t - b)(x - a) & \text{if } x \le t \le b. \end{cases}$$

Notice that the representation (6.93) was obtained by a different approach in [1].

Theorem 393 (Dragomir, 2010, [3]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ and let n be an integer with $n \ge 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then we have the representation*

$$f(A) = \frac{1}{M-m}\left[f(m)(M1_H - A) + f(M)(A - m1_H)\right] \tag{6.94}$$

$$+ \frac{(M1_H - A)(A - m1_H)}{M - m}$$

$$\times \sum_{k=1}^n \frac{1}{k!}\left\{f^{(k)}(m)(A - m1_H)^{k-1} + (-1)^k f^{(k)}(M)(M1_H - A)^{k-1}\right\}$$

$$+ T_n(f, m, M),$$

where the remainder $T_n(f, m, M)$ is given by

$$T_n(f, m, M) := \frac{1}{(M-m)\,n!}\int_{m-0}^M K_n(m, M, f; \lambda)\, dE_\lambda \tag{6.95}$$

and the kernel $K_n(m, M, f; \cdot)$ has the representation

$$K_n(m, M, f; \lambda) := (M - \lambda) \left(\int_m^\lambda (\lambda - t)^n \, d \left(f^{(n)}(t) \right) \right) \tag{6.96}$$

$$+ (-1)^{n+1} (\lambda - m) \left(\int_\lambda^M (t - \lambda)^n \, d \left(f^{(n)}(t) \right) \right)$$

for $\lambda \in [m, M]$.

Proof. Utilising Lemma 391 we have the representation

$$f(\lambda) = \frac{1}{M - m} \left[(M - \lambda) f(m) + (\lambda - m) f(M) \right] \tag{6.97}$$

$$+ \frac{(M - \lambda)(\lambda - m)}{M - m}$$

$$\times \sum_{k=1}^n \frac{1}{k!} \left\{ (\lambda - m)^{k-1} f^{(k)}(m) + (-1)^k (M - \lambda)^{k-1} f^{(k)}(M) \right\}$$

$$+ \frac{1}{(M - m) \, n!} \left[(M - \lambda) \int_m^\lambda (\lambda - t)^n \, d \left(f^{(n)}(t) \right) \right.$$

$$\left. + (-1)^{n+1} (\lambda - m) \int_\lambda^M (t - \lambda)^n \, d \left(f^{(n)}(t) \right) \right],$$

for any $\lambda \in [m, M]$.

If we integrate (6.97) in the Riemann-Stieltjes sense on the interval $[m, M]$ with the integrator E_λ, then we get

$$\int_{m-0}^M f(\lambda) \, dE_\lambda \tag{6.98}$$

$$= \frac{1}{M - m} \int_{m-0}^M \left[(M - \lambda) f(m) + (\lambda - m) f(M) \right] dE_\lambda$$

$$+ \int_{m-0}^M \frac{(M - \lambda)(\lambda - m)}{M - m} \sum_{k=1}^n \frac{1}{k!} \left\{ (\lambda - m)^{k-1} f^{(k)}(m) \right.$$

$$\left. + (-1)^k (M - \lambda)^{k-1} f^{(k)}(M) \right\} dE_\lambda + \frac{1}{(M - m) \, n!}$$

$$\times \left[\int_{m-0}^M (M - \lambda) \left(\int_m^\lambda (\lambda - t)^n \, d \left(f^{(n)}(t) \right) \right) dE_\lambda \right.$$

$$\left. + (-1)^{n+1} \int_{m-0}^M (\lambda - m) \left(\int_\lambda^M (t - \lambda)^n \, d \left(f^{(n)}(t) \right) \right) dE_\lambda \right].$$

Now, on making use of the spectral representation theorem we deduce from (6.98) the equality (6.85) with the remainder representation (6.86). ∎

Remark 394 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family. In the case when the function f is continuous and of bounded variation on $[m, M]$, then we get the representation*

$$f(A) = \frac{1}{M-m} [f(m)(M1_H - A) + f(M)(A - m1_H)] \tag{6.99}$$

$$+ \frac{1}{(M-m)} \left[\int_{m-0}^{M} (M-\lambda)[f(\lambda) - f(m)] dE_\lambda \right.$$

$$\left. - \int_{m-0}^{M} (\lambda - m)[f(M) - f(\lambda)] dE_\lambda \right].$$

Also, if the derivative f' is of bounded variation, then we have the representation

$$f(A) = \frac{1}{M-m} [f(m)(M1_H - A) + f(M)(A - m1_H)] \tag{6.100}$$

$$+ \frac{1}{(M-m)} \left[\int_{m-0}^{M} (M-\lambda) \left(\int_m^\lambda (\lambda - t) d(f'(t)) \right) dE_\lambda \right.$$

$$\left. + \int_{m-0}^{M} (\lambda - m) \left(\int_\lambda^M (t-\lambda) d(f'(t)) \right) dE_\lambda \right].$$

Example 395 *a. Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family. If we consider the exponential function, then we get from (6.94) and (6.95) that*

$$e^A = \frac{1}{M-m} [e^m(M1_H - A) + e^M(A - m1_H)] \tag{6.101}$$

$$+ \frac{(M1_H - A)(A - m1_H)}{M-m}$$

$$\times \sum_{k=1}^n \frac{1}{k!} \left\{ e^m(A - m1_H)^{k-1} + (-1)^k e^M(M1_H - A)^{k-1} \right\}$$

$$+ \frac{1}{(M-m)n!} \times \left[\int_{m-0}^{M} (M-\lambda) \left(\int_m^\lambda (\lambda - t)^n e^t dt \right) dE_\lambda \right.$$

$$\left. + (-1)^{n+1} \int_{m-0}^{M} (\lambda - m) \left(\int_\lambda^M (t-\lambda)^n e^t dt \right) dE_\lambda \right].$$

b. If A is a positive definite selfadjoint operator with the spectrum $Sp(A) \subseteq [m, M] \subset (0, \infty)$ and $\{E_\lambda\}_\lambda$ is its spectral family, then we have the repre-

sentation

$$\ln A = \frac{1}{M-m} \left[(M1_H - A) \ln m + (A - m1_H) \ln M \right] \qquad (6.102)$$

$$+ \frac{(M1_H - A)(A - m1_H)}{M-m}$$

$$\times \sum_{k=1}^{n} \frac{1}{k} \left\{ (-1)^{k-1} \frac{(A - m1_H)^{k-1}}{m^k} - \frac{(M1_H - A)^{k-1}}{M^k} \right\}$$

$$+ \frac{1}{(M-m)} \left[(-1)^n \int_{m-0}^{M} (M - \lambda) \left(\int_{m}^{\lambda} \frac{(\lambda - t)^n}{t^{n+1}} dt \right) dE_\lambda \right.$$

$$\left. - \int_{m-0}^{M} (\lambda - m) \left(\int_{\lambda}^{M} \frac{(t - \lambda)^n}{t^{n+1}} dt \right) dE_\lambda \right].$$

The case of functions for which the n-th derivative $f^{(n)}$ is absolutely continuous is of interest for applications. In this case the remainder can be represented as follows:

Theorem 396 (Dragomir, 2010, [3]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is absolutely continuous on the interval $[m, M]$, then we have the representation (6.94) where the remainder $T_n(f, m, M)$ is given by*

$$T_n(f, m, M) \qquad (6.103)$$

$$:= \frac{1}{(M-m)\, n!} \int_{m-0}^{M} W_n(m, M, f; \lambda)\, E_\lambda d\lambda$$

and the kernel $W_n(m, M, f; \cdot)$ has the representation

$$W_n(m, M, f; \lambda) \qquad (6.104)$$

$$:= (-1)^n \int_{m}^{\lambda} (\lambda - t)^{n-1} \left[nM + t - (n+1)\lambda \right] f^{(n+1)}(t)\, dt$$

$$- \int_{\lambda}^{M} (t - \lambda)^{n-1} \left[t + nm - (n+1)\lambda \right] f^{(n+1)}(t)\, dt$$

for $\lambda \in [m, M]$.

Proof. Observe that, by Leibnitz's rule for differentiation under the integral sign, we have

$$\frac{d}{d\lambda}\left[(M-\lambda)\left(\int_m^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt\right)\right] \tag{6.105}$$

$$= -\int_m^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt + (M-\lambda)\frac{d}{d\lambda}\left(\int_m^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt\right)$$

$$= -\int_m^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt + n(M-\lambda)\int_m^\lambda (\lambda-t)^{n-1} f^{(n+1)}(t)\,dt$$

$$= \int_m^\lambda (\lambda-t)^{n-1}[nM+t-(n+1)\lambda]\,f^{(n+1)}(t)\,dt$$

for any $\lambda \in [m, M]$.

Integrating by parts in the Riemann-Stieltjes integral we have

$$\int_{m-0}^M (M-\lambda)\left(\int_m^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt\right)dE_\lambda \tag{6.106}$$

$$= (M-\lambda)\left(\int_m^\lambda (\lambda-t)^n\,d\left(f^{(n)}(t)\right)\right)E_\lambda\Big|_{m-0}^M$$

$$- \int_{m-0}^M \left(\int_m^\lambda (\lambda-t)^{n-1}[nM+t-(n+1)\lambda]\,f^{(n+1)}(t)\,dt\right)E_\lambda\,d\lambda$$

$$= -\int_{m-0}^M \left(\int_m^\lambda (\lambda-t)^{n-1}[nM+t-(n+1)\lambda]\,f^{(n+1)}(t)\,dt\right)E_\lambda\,d\lambda.$$

By Leibnitz's rule we also have

$$\frac{d}{d\lambda}\left[(\lambda-m)\left(\int_\lambda^M (t-\lambda)^n f^{(n+1)}(t)\,dt\right)\right] \tag{6.107}$$

$$= \int_\lambda^M (t-\lambda)^n f^{(n+1)}(t)\,dt + (\lambda-m)\frac{d}{d\lambda}\left(\int_\lambda^M (t-\lambda)^n f^{(n+1)}(t)\,dt\right)$$

$$= \int_\lambda^M (t-\lambda)^n f^{(n+1)}(t)\,dt - n(\lambda-m)\int_\lambda^M (t-\lambda)^{n-1} f^{(n+1)}(t)\,dt$$

$$= \int_\lambda^M (t-\lambda)^{n-1}[t+nm-(n+1)\lambda]\,f^{(n+1)}(t)\,dt$$

for any $\lambda \in [m, M]$.

Utilising the integration by parts and (6.108) we get

$$\int_{m-0}^{M} (\lambda - m) \left(\int_{\lambda}^{M} (t - \lambda)^n f^{(n+1)}(t) \, dt \right) dE_\lambda \qquad (6.108)$$

$$= (\lambda - m) \left(\int_{\lambda}^{M} (t - \lambda)^n f^{(n+1)}(t) \, dt \right) E_\lambda \Big|_{m-0}^{M}$$

$$- \int_{m-0}^{M} \left(\int_{\lambda}^{M} (t - \lambda)^{n-1} [t + nm - (n+1)\lambda] f^{(n+1)}(t) \, dt \right) E_\lambda d\lambda$$

$$= - \int_{m-0}^{M} \left(\int_{\lambda}^{M} (t - \lambda)^{n-1} [t + nm - (n+1)\lambda] f^{(n+1)}(t) \, dt \right) E_\lambda d\lambda.$$

Finally, on utilizing the representation (6.95) for the remainder $T_n(f, m, M)$ and the equalities (6.106) and (6.108) we deduce (6.103). The details are omitted. ∎

Remark 397 *The case $n = 1$ provides the following equality*

$$f(A) = \frac{1}{M - m} \left[f(m)(M1_H - A) + f(M)(A - m1_H) \right] \qquad (6.109)$$

$$+ \frac{1}{(M - m)} \int_{m-0}^{M} W_1(m, M, f; \lambda) E_\lambda d\lambda,$$

where

$$W_1(m, M, f; \lambda) \qquad (6.110)$$

$$:= \int_{m}^{\lambda} (2\lambda - M - t) f''(t) \, dt + \int_{\lambda}^{M} (2\lambda - t - m) f''(t) \, dt$$

for $\lambda \in [m, M]$.

6.4.2   Error Bounds for $f^{(n)}$ of Bonded Variation

The following result that provides bounds for the absolute value of the kernel $K_n(m, M, f; \cdot)$ holds:

Lemma 398 (Dragomir, 2010, [3]) *Let I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$, let n be an integer with $n \geq 1$ and assume that $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ exists on the interval $[m, M]$.*

1. If $f^{(n)}$ is of bounded variation on $[m, M]$, then

$$|K_n(m, M, f; \lambda)| \qquad (6.111)$$

$$\leq (M - \lambda)(\lambda - m)^n \bigvee_m^\lambda \left(f^{(n)}\right) + (\lambda - m)(M - \lambda)^n \bigvee_\lambda^M \left(f^{(n)}\right)$$

$$\leq \frac{1}{4}(M - m)^2 \left[(\lambda - m)^{n-1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M - \lambda)^{n-1} \bigvee_\lambda^M \left(f^{(n)}\right)\right]$$

$$\leq \frac{1}{4}(M - m)^2 J_n(m, M; \lambda)$$

where

$$J_n(m, M; \lambda) \qquad (6.112)$$

$$:= \begin{cases} \left[\frac{1}{2}(M - m) + \left|\lambda - \frac{m+M}{2}\right|\right]^{n-1} \bigvee_m^M \left(f^{(n)}\right); \\[2ex] \left[(\lambda - m)^{p(n-1)} + (M - \lambda)^{p(n-1)}\right]^{1/p} \\ \quad \times \left[\left(\bigvee_m^\lambda \left(f^{(n)}\right)\right)^q + \left(\bigvee_\lambda^M \left(f^{(n)}\right)\right)^q\right]^{1/q} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \left[\frac{1}{2}\bigvee_m^M \left(f^{(n)}\right) + \frac{1}{2}\left|\bigvee_m^\lambda \left(f^{(n)}\right) - \bigvee_\lambda^M \left(f^{(n)}\right)\right|\right] \\ \quad \times \left[(\lambda - m)^{n-1} + (M - \lambda)^{n-1}\right] \end{cases}$$

and $\lambda \in [m, M]$.

2. If $\lambda \in (m, M)$ and $f^{(n)}$ is $L_{n,1,\lambda}$-Lipschitzian on $[m, \lambda]$ and $L_{n,2,\lambda}$-Lipschitzian on $[\lambda, M]$, then

$$|K_n(m, M, f; \lambda)| \qquad (6.113)$$

$$\leq \frac{1}{n+1}\left[L_{n,1,\lambda}(M - \lambda)(\lambda - m)^{n+1} + L_{n,2,\lambda}(\lambda - m)(M - \lambda)^{n+1}\right]$$

$$\leq \frac{1}{4(n+1)}\left[L_{n,1,\lambda}(\lambda - m)^n + L_{n,2,\lambda}(M - \lambda)^n\right]$$

$$\leq \frac{1}{4(n+1)}$$

$$\times \begin{cases} \left[(\lambda - m)^n + (M - \lambda)^n\right]\max\{L_{n,1,\lambda}, L_{n,2,\lambda}\} \\[2ex] \left[(\lambda - m)^{pn} + (M - \lambda)^{pn}\right]^{1/p}\left(L_{n,1,\lambda}^q + L_{n,2,\lambda}^q\right)^{1/q} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \left[\frac{1}{2}(M - m) + \left|\lambda - \frac{m+M}{2}\right|\right]^n (L_{n,1,\lambda} + L_{n,2,\lambda}) \end{cases}$$

and $\lambda \in [m, M]$.

In particular, if $f^{(n)}$ is L_n-Lipschitzian on $[m, M]$, then

$$|K_n(m, M, f; \lambda)| \tag{6.114}$$

$$\leq \frac{L_n}{n+1} \left[(M - \lambda)(\lambda - m)^{n+1} + (\lambda - m)(M - \lambda)^{n+1} \right]$$

$$\leq \frac{L_n (M - m)^2}{4(n+1)} [(\lambda - m)^n + (M - \lambda)^n]$$

and $\lambda \in [m, M]$.

3. If the function $f^{(n)}$ is monotonic nondecreasing on $[m, M]$, then

$$|K_n(m, M, f; \lambda)| \tag{6.115}$$

$$\leq (M - \lambda) \left[n \int_m^\lambda (\lambda - t)^{n-1} f^{(n)}(t)\, dt - (\lambda - m)^n f^{(n)}(m) \right]$$

$$+ (\lambda - m) \left[(M - \lambda)^n f^{(n)}(M) - n \int_\lambda^M (t - \lambda)^{n-1} f^{(n)}(t)\, dt \right]$$

$$\leq (M - \lambda)(\lambda - m)$$

$$\times \left[(\lambda - m)^{n-1} \left[f^{(n)}(\lambda) - f^{(n)}(m) \right] + (M - \lambda)^{n-1} \left[f^{(n)}(M) - f^{(n)}(\lambda) \right] \right]$$

$$\leq \frac{1}{4} (M - m)^2$$

$$\times \left[(\lambda - m)^{n-1} \left[f^{(n)}(\lambda) - f^{(n)}(m) \right] + (M - \lambda)^{n-1} \left[f^{(n)}(M) - f^{(n)}(\lambda) \right] \right]$$

$$\leq \frac{1}{4} (M - m)^2 T_n(m, M; \lambda)$$

where

$$T_n(m, M; \lambda) \tag{6.116}$$

$$:= \begin{cases} \left[\frac{1}{2}(M - m) + \left| \lambda - \frac{m+M}{2} \right| \right]^{n-1} \left[f^{(n)}(M) - f^{(n)}(m) \right]; \\[2mm] \left[(\lambda - m)^{p(n-1)} + (M - \lambda)^{p(n-1)} \right]^{1/p} \\ \times \left[\left(f^{(n)}(M) - f^{(n)}(\lambda) \right)^q + \left(f^{(n)}(\lambda) - f^{(n)}(m) \right)^q \right]^{1/q} \\ \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\[2mm] \left[\frac{1}{2} \left[f^{(n)}(M) - f^{(n)}(m) \right] + \left| f^{(n)}(\lambda) - \frac{f^{(n)}(M) + f^{(n)}(m)}{2} \right| \right] \\ \times \left[(\lambda - m)^{n-1} + (M - \lambda)^{n-1} \right]. \end{cases}$$

Proof. 1. It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral

$\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\, dv(t) \right| \le \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v),$$

(6.117)

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a,b]$.

Utilising the representation (6.96) and the property (6.117) we have successively

$|K_n(m, M, f; \lambda)|$

(6.118)

$$\le (M-\lambda) \left| \int_m^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t) \right) \right| + (\lambda - m) \left| \int_\lambda^M (t - \lambda)^n \, d\left(f^{(n)}(t) \right) \right|$$

$$\le (M-\lambda)(\lambda - m)^n \bigvee_m^\lambda \left(f^{(n)} \right) + (\lambda - m)(M - \lambda)^n \bigvee_\lambda^M \left(f^{(n)} \right)$$

$$= (M-\lambda)(\lambda - m) \left[(\lambda - m)^{n-1} \bigvee_m^\lambda \left(f^{(n)} \right) + (M - \lambda)^{n-1} \bigvee_\lambda^M \left(f^{(n)} \right) \right]$$

$$\le \frac{1}{4}(M - m)^2 \left[(\lambda - m)^{n-1} \bigvee_m^\lambda \left(f^{(n)} \right) + (M - \lambda)^{n-1} \bigvee_\lambda^M \left(f^{(n)} \right) \right]$$

$$\le \frac{1}{4}(M - m)^2 I_n(m, M; \lambda)$$

for any $\lambda \in [m, M]$.

By Hölder's inequality we also have

$$I_n(m, M; \lambda)$$

(6.119)

$$\le \begin{cases} \left[\frac{1}{2}(M-m) + \left| \lambda - \frac{m+M}{2} \right| \right]^{n-1} \bigvee_m^M \left(f^{(n)} \right); \\[2em] \left[(\lambda - m)^{p(n-1)} + (M - \lambda)^{p(n-1)} \right]^{1/p} \\[1em] \quad \times \left[\left(\bigvee_m^\lambda \left(f^{(n)} \right) \right)^q + \left(\bigvee_\lambda^M \left(f^{(n)} \right) \right)^q \right]^{1/q} \\[0.5em] \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\[1em] \left[\frac{1}{2} \bigvee_m^M \left(f^{(n)} \right) + \frac{1}{2} \left| \bigvee_m^\lambda \left(f^{(n)} \right) - \bigvee_\lambda^M \left(f^{(n)} \right) \right| \right] \\[1em] \quad \times \left[(\lambda - m)^{n-1} + (M - \lambda)^{n-1} \right]. \end{cases}$$

for any $\lambda \in [m, M]$.

On making use of (6.118) and (6.119) we deduce (6.111).

2. We recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b] ,$$

then the Riemann-Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) \, dv(t) \right| \leq L \int_a^b |p(t)| \, dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have

$$|K_n(m, M, f; \lambda)| \tag{6.120}$$

$$\leq (M - \lambda) \left| \int_m^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right| + (\lambda - m) \left| \int_\lambda^M (t - \lambda)^n \, d\left(f^{(n)}(t)\right) \right|$$

$$\leq \frac{1}{n+1} \left[L_{n,1,\lambda} (M - \lambda)(\lambda - m)^{n+1} + L_{n,2,\lambda} (\lambda - m)(M - \lambda)^{n+1} \right]$$

$$= \frac{(M - \lambda)(\lambda - m)}{n+1} \left[L_{n,1,\lambda} (\lambda - m)^n + L_{n,2,\lambda} (M - \lambda)^n \right]$$

$$\leq \frac{(M - m)^2}{4(n+1)} \left[L_{n,1,\lambda} (\lambda - m)^n + L_{n,2,\lambda} (M - \lambda)^n \right]$$

$$\leq \frac{(M - m)^2}{4(n+1)}$$

$$\times \begin{cases} [(\lambda - m)^n + (M - \lambda)^n] \max \{L_{n,1,\lambda}, L_{n,2,\lambda}\} \\ \\ [(\lambda - m)^{pn} + (M - \lambda)^{pn}]^{1/p} \left(L_{n,1,\lambda}^q + L_{n,2,\lambda}^q \right)^{1/q} \\ \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}(M - m) + \left| \lambda - \frac{m+M}{2} \right| \right]^n (L_{n,1,\lambda} + L_{n,2,\lambda}) \end{cases}$$

which prove the desired result (6.114).

3. From the theory of Riemann-Stieltjes integral is well known that if $p : [a, b] \to \mathbb{C}$ is continuous and $v : [a, b] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t) \, dv(t)$ and $\int_a^b |p(t)| \, dv(t)$ exist and

$$\left| \int_a^b p(t) \, dv(t) \right| \leq \int_a^b |p(t)| \, dv(t) \leq \max_{t \in [a,b]} |p(t)| [v(b) - v(a)]. \tag{6.121}$$

By utilizing this property, we have

$$|K_n(m, M, f; \lambda)| \tag{6.122}$$

$$\leq (M - \lambda) \left| \int_m^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right| + (\lambda - m) \left| \int_\lambda^M (t - \lambda)^n \, d\left(f^{(n)}(t)\right) \right|$$

$$\leq (M - \lambda) \int_m^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) + (\lambda - m) \int_\lambda^M (t - \lambda)^n \, d\left(f^{(n)}(t)\right)$$

$$= H_n(m, M; \lambda)$$

By the second part of (6.121) we also have that

$$H_n(m, M; \lambda) \tag{6.123}$$

$$\leq (M - \lambda)(\lambda - m)^n \left[f^{(n)}(\lambda) - f^{(n)}(m)\right]$$

$$+ (\lambda - m)(M - \lambda)^n \left[f^{(n)}(M) - f^{(n)}(\lambda)\right]$$

$$= (M - \lambda)(\lambda - m)$$

$$\times \left[(\lambda - m)^{n-1} \left[f^{(n)}(\lambda) - f^{(n)}(m)\right] + (M - \lambda)^{n-1} \left[f^{(n)}(M) - f^{(n)}(\lambda)\right]\right]$$

$$\leq \frac{1}{4}(M - m)^2$$

$$\times \left[(\lambda - m)^{n-1} \left[f^{(n)}(\lambda) - f^{(n)}(m)\right] + (M - \lambda)^{n-1} \left[f^{(n)}(M) - f^{(n)}(\lambda)\right]\right]$$

$$= \frac{1}{4}(M - m)^2 L_n(m, M; \lambda)$$

with

$$L_n(m, M; \lambda) \tag{6.124}$$

$$\leq \begin{cases} \left[\frac{1}{2}(M - m) + \left|\lambda - \frac{m+M}{2}\right|\right]^{n-1} \left[f^{(n)}(M) - f^{(n)}(m)\right]; \\[2mm] \left[(\lambda - m)^{p(n-1)} + (M - \lambda)^{p(n-1)}\right]^{1/p} \\ \times \left[\left(f^{(n)}(M) - f^{(n)}(\lambda)\right)^q + \left(f^{(n)}(\lambda) - f^{(n)}(m)\right)^q\right]^{1/q} \\ \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\[2mm] \left[\frac{1}{2}\left[f^{(n)}(M) - f^{(n)}(m)\right] + \left|f^{(n)}(\lambda) - \frac{f^{(n)}(M) + f^{(n)}(m)}{2}\right|\right] \\ \times \left[(\lambda - m)^{n-1} + (M - \lambda)^{n-1}\right]. \end{cases}$$

Integrating by parts we have

$$H_n \left(m, M; \lambda \right) \tag{6.125}$$

$$= \left(M - \lambda \right) \int_m^\lambda \left(\lambda - t \right)^n d \left(f^{(n)} \left(t \right) \right) + \left(\lambda - m \right) \int_\lambda^M \left(t - \lambda \right)^n d \left(f^{(n)} \left(t \right) \right)$$

$$= \left(M - \lambda \right) \left[\left(\lambda - t \right)^n f^{(n)} \left(t \right) \Big|_m^\lambda + n \int_m^\lambda \left(\lambda - t \right)^{n-1} f^{(n)} \left(t \right) dt \right]$$

$$+ \left(\lambda - m \right) \left[\left(t - \lambda \right)^n f^{(n)} \left(t \right) \Big|_\lambda^M - n \int_\lambda^M \left(t - \lambda \right)^{n-1} f^{(n)} \left(t \right) dt \right]$$

$$= \left(M - \lambda \right) \left[n \int_m^\lambda \left(\lambda - t \right)^{n-1} f^{(n)} \left(t \right) dt - \left(\lambda - m \right)^n f^{(n)} \left(m \right) \right]$$

$$+ \left(\lambda - m \right) \left[\left(M - \lambda \right)^n f^{(n)} \left(M \right) - n \int_\lambda^M \left(t - \lambda \right)^{n-1} f^{(n)} \left(t \right) dt \right].$$

On making use of (6.122)-(6.125) we deduce the desired result (6.115). ∎

On making use of the bounds for the kernel $K_n \left(m, M, f; \cdot \right)$ provided above, we can establish the following error estimates for the remainder $T_n \left(f, m, M \right)$ in the representation formula (6.94).

Theorem 399 (Dragomir, 2010, [3]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp \left(A \right) \subseteq \left[m, M \right]$ for some real numbers $m < M$, $\{ E_\lambda \}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $\left[m, M \right] \subset \mathring{I}$ and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $\left[m, M \right]$, then we have the representation*

$$\langle f \left(A \right) x, y \rangle \tag{6.126}$$

$$= \frac{1}{M - m} \left[f \left(m \right) \langle \left(M 1_H - A \right) x, y \rangle + f \left(M \right) \langle \left(A - m 1_H \right) x, y \rangle \right]$$

$$+ \frac{1}{M - m}$$

$$\times \left\{ \sum_{k=1}^n \frac{1}{k!} f^{(k)} \left(m \right) \left\langle \left(M 1_H - A \right) \left(A - m 1_H \right)^k x, y \right\rangle \right.$$

$$\left. + \sum_{k=1}^n \frac{1}{k!} \left(-1 \right)^k f^{(k)} \left(M \right) \left\langle \left(A - m 1_H \right) \left(M 1_H - A \right)^k x, y \right\rangle \right\}$$

$$+ T_n \left(f, m, M; x, y \right),$$

where the remainder $T_n \left(f, m, M; x, y \right)$ is given by

$$T_n \left(f, m, M; x, y \right) := \frac{1}{\left(M - m \right) n!} \int_{m-0}^M K_n \left(m, M, f; \lambda \right) d \langle E_\lambda x, y \rangle \tag{6.127}$$

and the kernel $K_n(m, M, f; \cdot)$ has the representation (6.96).

Moreover, we have the error estimate

$$|T_n(f, m, M; x, y)| \tag{6.128}$$

$$\leq \frac{1}{4n!}(M - m)\bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right)$$

$$\times \max_{\lambda \in [m,M]} \left[(\lambda - m)^{n-1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M - \lambda)^{n-1} \bigvee_\lambda^M \left(f^{(n)}\right)\right]$$

$$\leq \frac{1}{4n!}(M - m)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right)$$

$$\leq \frac{1}{4n!}(M - m)^n \bigvee_m^M \left(f^{(n)}\right) \|x\| \|y\|$$

for any $x, y \in H$.

Proof. The identity (6.126) with the remainder representation (6.127) follows from (6.94) and (6.95).

Now, on utilizing the property (6.117) for the Riemann-Stieltjes integral we deduce from (6.127) that

$$|T_n(f, m, M; x, y)| \tag{6.129}$$

$$\leq \frac{1}{(M - m)n!} \max_{\lambda \in [m,M]} |K_n(m, M, f; \lambda)| \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right)$$

for any $x, y \in H$.

Further, by (6.111) and (6.112) we have the bounds

$$|K_n(m, M, f; \lambda)| \tag{6.130}$$

$$\leq \frac{1}{4}(M - m)^2 \left[(\lambda - m)^{n-1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M - \lambda)^{n-1} \bigvee_\lambda^M \left(f^{(n)}\right)\right]$$

$$\leq \frac{1}{4}(M - m)^2 \left[\frac{1}{2}(M - m) + \left|\lambda - \frac{m + M}{2}\right|\right]^{n-1} \bigvee_m^M \left(f^{(n)}\right);$$

for any $\lambda \in [m, M]$.

Taking the maximum over $\lambda \in [m, M]$ in (6.130) we deduce the first and the second inequalities in (6.128).

The last part follows by the Total Variation Schwarz's inequality and we omit the details. ∎

Corollary 400 (Dragomir, 2010, [3]) *With the assumptions from The-orem 399 and if $f^{(n)}$ is L_n-Lipschitzian on $[m, M]$, then*

$$|T_n (f, m, M; x, y)| \tag{6.131}$$

$$\leq \frac{1}{(n+1)! \, (M-m)} L_n \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\times \max_{\lambda \in [m,M]} \left[(M-\lambda)(\lambda-m)^{n+1} + (\lambda-m)(M-\lambda)^{n+1} \right]$$

$$\leq \frac{1}{4 \, (n+1)!} (M-m)^{n+1} L_n \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$

$$\leq \frac{1}{4 \, (n+1)!} (M-m)^{n+1} L_n \, \|x\| \, \|y\|$$

for any $x, y \in H$.

6.4.3   Error Bounds for $f^{(n)}$ Absolutely Continuous

The following result that provides bounds for the absolute value of the kernel $W_n \, (m, M, f; \cdot)$ holds:

Lemma 401 (Dragomir, 2010, [3]) *Let I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$, let n be an integer with $n \geq 1$ and assume that $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is absolutely continuous on the interval $[m, M]$. Then we have the bound*

$$|W_n \, (m, M, f; \lambda)| \leq \sum_{i=1}^{4} B_n^{(i)} \, (m, M, f; \lambda) \tag{6.132}$$

where

$$B_n^{(1)} \, (m, M, f; \lambda) \tag{6.133}$$

$$:= n \, (M-\lambda) \int_m^\lambda (\lambda - t)^{n-1} \left| f^{(n+1)} (t) \right| dt \leq n \, (M-\lambda)$$

$$\times \begin{cases} \frac{1}{n} (\lambda - m)^n \left\| f^{(n+1)} \right\|_{[m,\lambda],\infty} & \text{if } f^{(n+1)} \in L_\infty \, [m, \lambda] \, ; \\[2mm] \frac{1}{[(n-1)p_1+1]^{1/p_1}} (\lambda - m)^{n-1+1/p_1} \left\| f^{(n+1)} \right\|_{[m,\lambda],q_1} \\ \text{if } f^{(n+1)} \in L_{q_1} \, [m, \lambda] \, , p_1 > 1, \frac{1}{p_1} + \frac{1}{q_1} = 1; \\[2mm] (\lambda - m)^{n-1} \left\| f^{(n+1)} \right\|_{[m,\lambda],1} \, ; \end{cases}$$

$$B_n^{(2)}(m, M, f; \lambda) \tag{6.134}$$

$$:= \int_m^\lambda (\lambda - t)^n \left| f^{(n+1)}(t) \right| dt$$

$$\leq \begin{cases} \frac{1}{n+1} (\lambda - m)^{n+1} \left\| f^{(n+1)} \right\|_{[m,\lambda],\infty} & \text{if } f^{(n+1)} \in L_\infty[m, \lambda]; \\[2mm] \frac{1}{(np_2+1)^{1/p_2}} (\lambda - m)^{n+1/p_2} \left\| f^{(n+1)} \right\|_{[m,\lambda],q_2} & \text{if } f^{(n+1)} \in L_{q_2}[m, \lambda], \\ p_2 > 1, \frac{1}{p_2} + \frac{1}{q_2} = 1; \\[2mm] (\lambda - m)^n \left\| f^{(n+1)} \right\|_{[m,\lambda],1} \end{cases}$$

$$B_n^{(3)}(m, M, f; \lambda) \tag{6.135}$$

$$:= \int_\lambda^M (t - \lambda)^n \left| f^{(n+1)}(t) \right| dt$$

$$\leq \begin{cases} \frac{1}{n+1} (M - \lambda)^{n+1} \left\| f^{(n+1)} \right\|_{[\lambda,M],\infty} & \text{if } f^{(n+1)} \in L_\infty[\lambda, M]; \\[2mm] \frac{1}{(np_3+1)^{1/p_3}} (M - \lambda)^{n+1/p_3} \left\| f^{(n+1)} \right\|_{[\lambda,M],q_3} & \text{if } f^{(n+1)} \in L_{q_3}[\lambda, M], \\ p_3 > 1, \frac{1}{p_3} + \frac{1}{q_3} = 1; \\[2mm] (M - \lambda)^n \left\| f^{(n+1)} \right\|_{[\lambda,M],1} \end{cases}$$

and

$$B_n^{(4)}(m, M, f; \lambda) \tag{6.136}$$

$$:= n(\lambda - m) \int_\lambda^M (t - \lambda)^{n-1} \left| f^{(n+1)}(t) \right| dt \leq n(\lambda - m)$$

$$\times \begin{cases} \frac{1}{n} (M - \lambda)^n \left\| f^{(n+1)} \right\|_{[\lambda,M],\infty} & \text{if } f^{(n+1)} \in L_\infty[\lambda, M]; \\[2mm] \frac{1}{[(n-1)p_4+1]^{1/p_4}} (M - \lambda)^{n-1+1/p_4} \left\| f^{(n+1)} \right\|_{[\lambda,M],q_4} \\ \text{if } f^{(n+1)} \in L_{q_1}[\lambda, M], p_4 > 1, \frac{1}{p_4} + \frac{1}{q_4} = 1; \\[2mm] (M - \lambda)^{n-1} \left\| f^{(n+1)} \right\|_{[m,\lambda],1}; \end{cases}$$

for any $\lambda \in [m, M]$, *where the Lebesgue norms* $\|\cdot\|_{[a,b],p}$ *are defined by*

$$\|g\|_{[a,b],p} := \begin{cases} \left(\int_a^b |g(t)|^p \, dt \right)^{1/p} & \text{if } g \in L_p[a, b], p \geq 1 \\[3mm] ess \sup_{t \in [a,b]} |g(t)| & \text{if } g \in L_\infty[a, b]. \end{cases}$$

Proof. From (6.104) we have

$$|W_n (m, M, f; \lambda)| \tag{6.137}$$

$$\leq \left| \int_m^\lambda (\lambda - t)^{n-1} [nM + t - (n+1)\lambda] f^{(n+1)}(t) \, dt \right|$$

$$+ \left| \int_\lambda^M (t - \lambda)^{n-1} [t + nm - (n+1)\lambda] f^{(n+1)}(t) \, dt \right|$$

$$\leq \int_m^\lambda (\lambda - t)^{n-1} |nM + t - (n+1)\lambda| \left| f^{(n+1)}(t) \right| dt$$

$$+ \int_\lambda^M (t - \lambda)^{n-1} |t + nm - (n+1)\lambda| \left| f^{(n+1)}(t) \right| dt$$

$$\leq \int_m^\lambda (\lambda - t)^{n-1} [n(M - \lambda) + (\lambda - t)] \left| f^{(n+1)}(t) \right| dt$$

$$+ \int_\lambda^M (t - \lambda)^{n-1} [(t - \lambda) + n(\lambda - m)] \left| f^{(n+1)}(t) \right| dt$$

$$= \sum_{i=1}^4 B_n^{(i)} (m, M, f; \lambda)$$

for any $\lambda \in [m, M]$, which proves (6.132).

The other bounds follows by Hölder's integral inequality and the details are omitted. ∎

Remark 402 *It is obvious that the inequalities (6.132)-(6.136) can produce 12 different bounds for $|W_n (m, M, f; \lambda)|$. However, we mention here only the case when $f^{(n+1)} \in L_\infty [\lambda, M]$, namely*

$$|W_n (m, M, f; \lambda)| \tag{6.138}$$

$$\leq (M - \lambda)(\lambda - m)^n \left\| f^{(n+1)} \right\|_{[m,\lambda],\infty} + \frac{1}{n+1} (\lambda - m)^{n+1} \left\| f^{(n+1)} \right\|_{[m,\lambda],\infty}$$

$$+ \frac{1}{n+1} (M - \lambda)^{n+1} \left\| f^{(n+1)} \right\|_{[\lambda,M],\infty} + (\lambda - m)(M - \lambda)^n \left\| f^{(n+1)} \right\|_{[\lambda,M],\infty}$$

$$\leq [(M - \lambda)(\lambda - m)^n + (\lambda - m)(M - \lambda)^n$$

$$+ \frac{1}{n+1} (\lambda - m)^{n+1} + \frac{1}{n+1} (M - \lambda)^{n+1}] \left\| f^{(n+1)} \right\|_{[m,M],\infty}$$

for any $\lambda \in [m, M]$.

Finally, we can state the following result as well:

Theorem 403 (Dragomir, 2010, [3]) *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on*

$\mathbb{R}$ *with* $[m, M] \subset \overset{\circ}{I}$ *and let* n *be an integer with* $n \geq 1$. *If* $f : I \to \mathbb{C}$ *is such that the n-th derivative* $f^{(n)}$ *is absolutely continuous on the interval* $[m, M]$, *then we have the representation (6.126) where the remainder* $T_n(f, m, M; x, y)$ *is given by*

$$T_n(f, m, M; x, y) \tag{6.139}$$

$$:= \frac{1}{(M-m)\, n!} \int_{m-0}^{M} W_n(m, M, f; \lambda) \langle E_\lambda x, y \rangle \, d\lambda$$

and the kernel $W_n(m, M, f; \cdot)$ *has the representation (6.104).*
   We also have the error bounds

$$|T_n(f, m, M; x, y)| \tag{6.140}$$

$$\leq \frac{1}{(M-m)\, n!} \int_{m-0}^{M} |W_n(m, M, f; \lambda)| \, |\langle E_\lambda x, y \rangle| \, d\lambda$$

$$\leq \frac{1}{(M-m)\, n!} \int_{m-0}^{M} |W_n(m, M, f; \lambda)| \, \langle E_\lambda x, x \rangle^{1/2} \, \langle E_\lambda y, y \rangle^{1/2} \, d\lambda$$

$$\leq \frac{1}{(M-m)\, n!} \, \|x\| \, \|y\| \int_{m}^{M} |W_n(m, M, f; \lambda)| \, d\lambda$$

for any $x, y \in H$.

Remark 404 *On making use of Lemma 401 one can produce further bounds. However, the details are left to the interested reader.*

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